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CONTINUOUS AND DISCONTINUOUS LINEAR APPROXIMATION OF THE WINDOW SPECTRUM BY LEAST SQUARES METHOD

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Abstract

This paper presents the general solution of the least-squares approximation of the frequency characteristic of the data window by linear functions combined with zero padding technique. The approximation characteristic can be discontinuous or continuous, what depends on the value of one approximation parameter. The approximation solution has an analytical form and therefore the results have universal character. The paper presents derived formulas, analysis of approximation accuracy, the exemplary characteristics and conclusions, which confirm high accuracy of the approximation. The presented solution is applicable to estimating methods, like the LIDFT method, visualizations, etc.

Keywords: data window, spectrum approximation, interpolated DFT, LIDFT, spectrum estimation, zero padding.

1. Introduction

The least-squares approximation of the data window frequency characteristics with appropriate linear functions is used in the linear interpolation of the discrete Fourier transform (LIDFT) [1-3] and the zero padding can also be used with this approximation to increase approximation accuracy [4]. Linear approximation of the spectrum allows for linearizing relationships to determine component frequencies and this feature of the LIDFT method differs it from other methods of signal estimation [5-13].

The piece-wise linear DFT approximation is used in [14] but in the simple form, not by least squares method. In the methods described in [1-4] the approximation function is discontinuous, which can be inadvisable in some applications.

The paper removes discontinuity from the approximation function and is organized as follows. Symbols and basic equations are defined in Section 2. In section 3 the approximation from [4], with zero padding technique, is presented in more general form than in previous papers [1-4]. Afterwards the alternative approximation is analysed in Section 4. Section 5 describes how the formulas for continuous approximation function are obtained and Section 6 presents the exemplary characteristics for a triangle data window.

2. The data window frequency characteristic

The data window frequency characteristic $W(\lambda)$ is given by equation:

$$W(\lambda) = \sum_{n=n_0}^{n_0+N-1} w_n e^{-j2\pi n\lambda/N}, \tag{1}$$

where: $\lambda = fNT$ is the normalized frequency (in [bins]), f is the frequency in [Hz], T is the

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sampling period, N is the length of data window, which is defined by values w_n and n_0 denotes the first value of the sample index. Most often $n_0 = 0$, what gives the discrete-time Fourier transform (DtFT) of the data window or $n_0 = -N/2$ what gives shifted DtFT of the data window. In this paper the shifted DtFT is used ($n_0 = -N/2$) because for a symmetric data window, the $W(\lambda)$ has only real values. The non-shifted standard DtFT is obtained from $W(\lambda)$ for $n_0 = -N/2$ (and *vice-versa*) by simple formula:

$$W(\lambda) = \sum_{n=-N/2}^{N/2-1} w_n e^{-\mathrm{j}2\pi n\lambda/N} = e^{j\pi\lambda} \sum_{n=0}^{N-1} w_n e^{-\mathrm{j}2\pi n\lambda/N} . \tag{2}$$

The zero padding technique involves computing (2) for the M-element series $\{w_{-N/2}, \ldots, w_{N/2-1}, 0, \ldots, 0\}_{NR}$ instead of the series $\{w_{-N/2}, \ldots, w_{N/2-1}\}_N$. Such a "$R$-times" zero padding technique (where $M = NR$) takes the following form from (2):

$$W(\lambda/R) = \sum_{n=-N/2}^{N/2-1} w_n e^{-\mathrm{j}2\pi n\lambda/M} . \tag{3}$$

The values $W(\lambda/R)$ for integer values of λ are easy to obtain by using the FFT algorithm and (2)-(3):

$$\sum_{n=-N/2}^{N/2-1} w_n e^{-j2\pi nk/M} = e^{j\pi k/R} \mathrm{FFT}_k \{w_n\}_{NR} , \tag{4}$$

where: $\{w_n\}_{NR}$ is the zero padded data window:

$$\{w_n\}_{NR} = \{\{w_n\}_N , \{0,\ldots,0\}_{N(R-1)}\} \tag{5}$$

and $M = NR$ is the natural power of 2 for radix-2 FFT algorithm.

The window spectrum $W(\lambda)$ defined by (2) is approximated in the LIDFT method by function $\hat{W}(\lambda)$ and in the least squares method the quality of this approximation is given by the mean square approximation error Q:

$$Q = \int_{-N/2}^{N/2} | W(\lambda) - \hat{W}(\lambda) |^2 \, d\lambda . \tag{6}$$

This error has a different form in Sections 3-5, what gives different formulas for the parameters of the function $\hat{W}(\lambda)$.

3. The approximation by linear functions used in the LIDFT method

For data window and the zero padding technique, the LIDFT method assumes an approximation of the frequency characteristic $W(\lambda)$ from (2) using linear functions, as shown in Fig. 1a [4]. The linear functions $\hat{W}_k(\lambda) = a_k \lambda + b_k$ approximate the window characteristic $W(\lambda)$ for $\lambda \in [k/R, (k+1)/R]$. The error Q from (6) is defined as:

$$Q = \sum_{k=-NR/2}^{NR/2-1} \int_{k/R}^{(k+1)/R} \left| W(\lambda) - \hat{W}_k(\lambda) \right|^2 d\lambda \tag{7}$$

and is described in the matrix form (on the base of (A7) of Appendix A) by:

$$Q = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w} + (\mathbf{c} - \mathbf{z})^{\mathrm{H}} \mathbf{A}(\mathbf{c} - \mathbf{z}) - \mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{z} = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w} + \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{c} - \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{z} - \mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{c} , \tag{8}$$

where:

$$\mathbf{w} = [w_{-N/2},\ldots, w_{N/2-1}]^{\mathrm{T}} , \tag{9}$$

$$\mathbf{A} = R \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 3 \cdot \mathbf{I} \end{bmatrix}, \tag{10}$$

$$\mathbf{c} = [\mathbf{c}'^{\mathrm{T}} \quad \mathbf{c}''^{\mathrm{T}}]^{\mathrm{T}}, \tag{11}$$

$$\mathbf{c}' = [c'_{-M/2}, \dots, c'_{M/2-1}]^{\mathrm{T}}, \qquad c'_k = \frac{1}{R}\left(a_k \frac{k+1/2}{R} + b_k\right), \tag{12}$$

$$\mathbf{c}'' = [c''_{-M/2}, \dots, c''_{M/2-1}]^{\mathrm{T}}, \qquad c''_k = \frac{a_k}{6R^2}, \tag{13}$$

$$\mathbf{z} = [\mathbf{z}'^{\mathrm{T}} \quad \mathbf{z}''^{\mathrm{T}}]^{\mathrm{T}}, \tag{14}$$

$$\mathbf{z}' = [z'_{-M/2}, \dots, z'_{M/2-1}]^{\mathrm{T}}, \qquad z'_k = \int_{k/R}^{(k+1)/R} W(\lambda) d\lambda, \tag{15}$$

$$\mathbf{z}'' = [z''_{-M/2}, \dots, z''_{M/2-1}]^{\mathrm{T}}, \qquad z''_k = 2R \int_{k/R}^{(k+1)/R} \left(\lambda - \frac{k+1/2}{R}\right) W(\lambda) d\lambda. \tag{16}$$

The coefficients z'_k, z''_k from (15)-(16) are given, by use of (2), as:

$$z'_k = \sum_{n=-N/2}^{N/2-1} w_n \int_{k/R}^{(k+1)/R} e^{-j2\pi n\lambda/N} d\lambda, \qquad z''_k = 2R \sum_{n=-N/2}^{N/2-1} w_n \int_{k/R}^{(k+1)/R} \left(\lambda - \frac{k+1/2}{R}\right) e^{-j2\pi n\lambda/N} d\lambda. \tag{17}$$

The integrals in (17) are given by:

$$\int_{k/R}^{(k+1)/R} e^{-j2\pi n\lambda/N} d\lambda = \frac{\operatorname{sinc} x_n}{R} e^{-jx_n} e^{-j2\pi nk/M}, \tag{18}$$

$$\int_{k/R}^{(k+1)/R} \left(\lambda - \frac{k+1/2}{R}\right) e^{-j2\pi n\lambda/N} d\lambda = j \frac{\operatorname{sinc}' x_n}{2R^2} \cdot e^{-jx_n} e^{-j2\pi nk/M}, \tag{19}$$

where:

$$x_n = \frac{\pi n}{M}, \qquad M = NR, \qquad n = -N/2, \dots, N/2-1, \tag{20}$$

$$\operatorname{sinc} x_n = \frac{\sin x_n}{x_n}, \qquad \operatorname{sinc}' x_n = \frac{d(\operatorname{sinc} x_n)}{dx_n} = \frac{\cos x_n - (\sin x_n)/x_n}{x_n}. \tag{21}$$

Taking into account (17)-(21), the matrices (15)-(16) are given by:

$$\mathbf{z}' = R^{-1} \mathbf{W}_R \cdot \mathbf{w}', \qquad \mathbf{z}'' = R^{-1} \mathbf{W}_R \cdot \mathbf{w}'', \tag{22}$$

where:

$$\mathbf{W}_R = [e^{-j2\pi nk/M}]_{M \times N}, \quad k = -M/2, \dots, M/2-1, \quad n = -N/2, \dots, N/2-1, \tag{23}$$

$$\mathbf{w}' = [w'_{-N/2}, \dots, w'_{N/2-1}]^{\mathrm{T}}, \quad w'_n = w_n (\operatorname{sinc} x_n) e^{-jx_n}, \tag{24}$$

$$\mathbf{w}'' = [w''_{-N/2}, \dots, w''_{N/2-1}]^{\mathrm{T}}, \quad w''_n = j w_n (\operatorname{sinc}' x_n) e^{-jx_n}. \tag{25}$$

Let us notice that:

$$\mathbf{W}_R^{\mathrm{H}} \mathbf{W}_R = M \cdot \mathbf{I}. \tag{26}$$

From (24) the matrix $\mathbf{z}$ (14) is given by:

$$\mathbf{z} = \frac{1}{R} \begin{bmatrix} \mathbf{W}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_R \end{bmatrix} \cdot \begin{bmatrix} \mathbf{w}' \\ \mathbf{w}'' \end{bmatrix}. \tag{27}$$

The task of determining the coefficients a_k, b_k of the approximating linear functions that minimize Q defined by (8) can be replaced by an equivalent task of determining the coefficients c'_k, c''_k from (12)-(13). They are obtained from:

$$\frac{dQ}{d\mathbf{c}} = 2\mathbf{A}(\mathbf{c}-\mathbf{z}) = \mathbf{0}, \tag{28}$$

which gives:

$$\mathbf{c} = \mathbf{z} \tag{29}$$

and then Q has minimum value determined on the basis of (8) and (29):

$$Q_{\min} = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w} - \mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{z}. \tag{30}$$

From (10), (24)-(27) the expression $\mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{z}$ from (30) is given by:

$$\mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{z} = N \cdot [(\mathbf{w}')^{\mathrm{H}}\mathbf{w}' + 3(\mathbf{w}'')^{\mathrm{H}}\mathbf{w}''] = N\sum_{n=-N/2}^{N/2-1} w_n^2[(\operatorname{sinc} x_n)^2 + 3(\operatorname{sinc}' x_n)^2] \tag{31}$$

and then (30) has the following form:

$$Q_{\min} = N\sum_{n=-N/2}^{N/2-1} c_n w_n^2, \qquad c_n = 1 - (\operatorname{sinc} x_n)^2 - 3(\operatorname{sinc}' x_n)^2. \tag{32}$$

Let us define the vectors $\mathbf{d}'$ and $\mathbf{d}''$ and coefficients d'_k, d''_k, based on the values of approximating function $\hat{W}_k(\lambda)$ for $\lambda = k/R$, $(k+1/2)/R$, $(k+1)/R$ (two extreme values and the median value of the range $[k/R, (k+1)/R]$) as follows:

$$\mathbf{d}' = [d'_{-M/2},\ldots,d'_{M/2-1}]^{\mathrm{T}}, \qquad d'_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) = a_k\frac{k+1/2}{R} + b_k, \tag{33}$$

$$\mathbf{d}'' = [d''_{-M/2},\ldots,d''_{M/2-1}]^{\mathrm{T}}, \qquad d''_k = \hat{W}_k\left(\frac{k+1}{R}\right) - \hat{W}_k\left(\frac{k}{R}\right) = \frac{a_k}{R}. \tag{34}$$

Using (11)–(13) they can be described as:

$$\mathbf{d} = [\mathbf{d}'^{\mathrm{T}} \quad \mathbf{d}''^{\mathrm{T}}]^{\mathrm{T}} = R\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & 6\cdot\mathbf{I}\end{bmatrix}\cdot\mathbf{c} \tag{35}$$

and taking into account (27), (29) one has:

$$\mathbf{d} = [\mathbf{d}'^{\mathrm{T}} \quad \mathbf{d}''^{\mathrm{T}}]^{\mathrm{T}} = \begin{bmatrix}\mathbf{W}_R & \mathbf{0}\\ 0 & \mathbf{W}_R\end{bmatrix}\cdot\begin{bmatrix}\mathbf{w}'\\ 6\cdot\mathbf{w}''\end{bmatrix} \tag{36}$$

or in non-matrix form by:

$$d'_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) = a_k\frac{k+1/2}{R} + b_k = \sum_{n=-N/2}^{N/2-1} w_n \operatorname{sinc} x_n e^{-jx_n}e^{-j2\pi nk/M}, \tag{37}$$

$$d''_k = \hat{W}_k\left(\frac{k+1}{R}\right) - \hat{W}_k\left(\frac{k}{R}\right) = \frac{a_k}{R} = j6\sum_{n=-N/2}^{N/2-1} w_n \operatorname{sinc}' x_n e^{-jx_n}e^{-j2\pi nk/M}. \tag{38}$$

The coefficients d'_k, d''_k can be calculated by FFT from (4).

The equations (36)-(38) allow the calculation of the parameters d'_k, d''_k of approximating linear functions, and hence their coefficients a_k, b_k can be calculated. The graphical interpretation of d'_k, d''_k is shown in Fig. 1b.

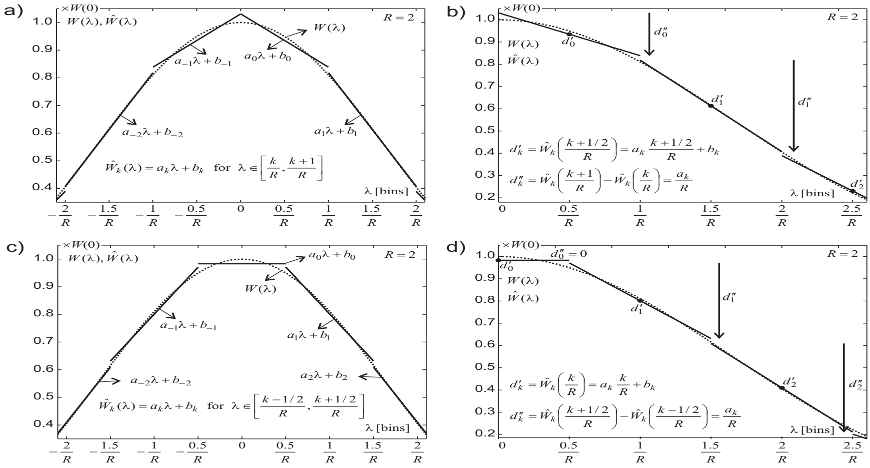

Fig. 1. Approximation of data window spectrum by linear functions: a,b) in Section 3; c,d) in Section 4

4. Alternative approximation by linear functions

The approximation alternative to the presented in Section 3 is defined by $\hat{W}_k(\lambda) = a_k \lambda + b_k$ for $\lambda \in$ [$(k$–$0.5)/R$, $(k$+$0.5)/R$] (Fig. 1c). The mean square error Q of such approximation is different from (7), i.e. it has a different range of integration:

$$Q = \sum_{k=-NR/2}^{NR/2-1} \int_{(k-1/2)/R}^{(k+1/2)/R} \left| W(\lambda) - \hat{W}_k(\lambda) \right|^2 d\lambda \tag{39}$$

and, on the basis of (A10) of Appendix A, is equal (8) where matrices $\mathbf{w}$, $\mathbf{A}$, $\mathbf{c}$, $\mathbf{c}''$, $\mathbf{z}$ are defined by (9)-(11), (13)-(14), and $\mathbf{c}'$, $\mathbf{z}'$, $\mathbf{z}''$ are defined by:

$$\mathbf{c}' = [c'_{-M/2},...,c'_{M/2-1}]^{\mathrm{T}}, \qquad c'_k = \frac{1}{R}\left(a_k \frac{k}{R} + b_k \right), \tag{40}$$

$$\mathbf{z}' = [z'_{-M/2},...,z'_{M/2-1}]^{\mathrm{T}}, \qquad z'_k = \int_{(k-1/2)/R}^{(k+1/2)/R} W(\lambda) d\lambda, \tag{41}$$

$$\mathbf{z}'' = [z''_{-M/2},...,z''_{M/2-1}]^{\mathrm{T}}, \qquad z''_k = 2R \int_{(k-1/2)/R}^{(k+1/2)/R} \left(\lambda - \frac{k}{R} \right) W(\lambda) d\lambda. \tag{42}$$

After taking into account (2), the coefficients z'_k, z''_k from (41)-(42) can be written as:

$$z'_k = \sum_{n=-N/2}^{N/2-1} w_n \int_{(k-1/2)/R}^{(k+1/2)/R} e^{-j2\pi n\lambda/N} d\lambda, \qquad z''_k = 2R \sum_{n=-N/2}^{N/2-1} w_n \int_{(k-1/2)/R}^{(k+1/2)/R} \left(\lambda - \frac{k}{R} \right) e^{-j2\pi n\lambda/N} d\lambda. \tag{43}$$

The integrals in both of these dependencies are given by:

$$\int_{(k-1/2)/R}^{(k+1/2)/R} e^{-j2\pi n\lambda/N} d\lambda = \frac{\operatorname{sinc} x_n}{R} \cdot e^{-j2\pi nk/M}, \qquad \int_{(k-1/2)/R}^{(k+1/2)/R} \left(\lambda - \frac{k}{R} \right) e^{-j2\pi n\lambda/N} d\lambda = j \frac{\operatorname{sinc}' x_n}{2R^2} \cdot e^{-j2\pi nk/M}, \tag{44}$$

where: x_n, sinc x_n, sinc$'x_n$ are defined by (20)-(21).

The matrices (41)-(42) can be written as in (22), i.e. $\mathbf{z}' = R^{-1}\mathbf{W}_R \cdot \mathbf{w}'$, $\mathbf{z}'' = R^{-1}\mathbf{W}_R \cdot \mathbf{w}''$, where $\mathbf{W}_R$ is defined by (23), and vectors $\mathbf{w}'$, $\mathbf{w}''$ are defined as:

$$\mathbf{w}' = [w'_{-N/2},...,w'_{N/2-1}]^{\mathrm{T}}, \quad w'_n = w_n(\mathrm{sinc}\; x_n), \tag{45}$$

$$\mathbf{w}'' = [w''_{-N/2},...,w''_{N/2-1}]^{\mathrm{T}}, \quad w''_n = jw_n(\mathrm{sinc'}\, x_n). \tag{46}$$

The task of determining the coefficients a_k, b_k (c'_k, c''_k) of the approximating linear functions that minimize Q defined by (39) is obtained by using equations (28)-(29), and similar transformations as in (30)-(31) give the same result for the mean square error of alternative approximation as the (32).

By defining the vectors $\mathbf{d}'$ and $\mathbf{d}''$ with coefficients d'_k, d''_k, based on the values of approximating function $\hat{W}_k(\lambda)$ for $\lambda = (k–1/2)/R$, k/R, $(k+1/2)/R$ (two extreme values and the median value of the range $[(k–1/2)/R, (k+1/2)/R]$) as follows:

$$\mathbf{d}' = [d'_{-M/2},...,d'_{M/2-1}]^{\mathrm{T}}, \qquad d'_k = \hat{W}_k\left(\frac{k}{R}\right) = a_k\frac{k}{R} + b_k, \tag{47}$$

$$\mathbf{d}'' = [d''_{-M/2},...,d''_{M/2-1}]^{\mathrm{T}}, \qquad d''_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) - \hat{W}_k\left(\frac{k-1/2}{R}\right) = \frac{a_k}{R}, \tag{48}$$

it is possible, after taking into account (11), (13), (40), (45)-(46), to express them in the matrix form (35)-(36) or in the non-matrix form by:

$$d'_k = \hat{W}_k\left(\frac{k}{R}\right) = a_k\frac{k}{R} + b_k = \sum_{n=-N/2}^{N/2-1} w_n \mathrm{sinc}\; x_n e^{-j2\pi nk/M}, \tag{49}$$

$$d''_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) - \hat{W}_k\left(\frac{k-1/2}{R}\right) = \frac{a_k}{R} = j6\sum_{n=-N/2}^{N/2-1} w_n \mathrm{sinc'}\, x_n e^{-j2\pi nk/M}. \tag{50}$$

The coefficients d'_k, d''_k can be calculated by FFT from (4).

The equations (49)-(50) allow the calculation of the parameters d'_k, d''_k of approximating linear functions, and hence their coefficients a_k, b_k can be calculated. The graphical interpretation of d'_k, d''_k is shown in Fig. 1d.

Both approximations from the Sections 3 and 4 shown in Figure 1 are discontinuous, which may be disadvantageous in some applications. Their modification presented in the next Section, minimizes this discontinuity.

5. Reduction of approximation discontinuity

There is a natural question whether it is possible to eliminate or at least reduce the approximation discontinuity at points of transition to the next line fitting (Fig. 1). To this aim, it is possible to minimize the error (7) taking into account, with the weight μ, an additional component which is the mean square error of discontinuity:

$$Q_1 = Q + \mu \sum_{k=-NR/2}^{NR/2-1} \left| \hat{W}_{k+1}\left(\frac{k+1}{R}\right) - \hat{W}_k\left(\frac{k+1}{R}\right) \right|^2 \tag{51}$$

or in the same way for the error (39):

$$Q_1 = Q + \mu \sum_{k=-NR/2}^{NR/2-1} \left| \hat{W}_{k+1}\left(\frac{k+1/2}{R}\right) - \hat{W}_k\left(\frac{k+1/2}{R}\right) \right|^2 . \tag{52}$$

In both cases (based on equation (A12) of Appendix A), the Q_1 is given by:

$$Q_1 = Q + \mu \mathbf{c}^{\mathrm{H}} \mathbf{A} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}] \mathbf{A}\mathbf{c} , \tag{53}$$

where: Q, $\mathbf{w}$, $\mathbf{A}$ are defined by (8)-(10) and $\mathbf{c}$, $\mathbf{z}$ are defined by (11)-(16) for the approximation from Section 3 and by (11), (13)-(14), (40)-(42) for the approximation from Section 4 and, additionally, the matrix $\mathbf{E}$ is a circulant matrix defined as:

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{54}$$

Minimization of error (53) is achieved for the condition:

$$\frac{dQ_1}{d\mathbf{c}} = 2\mathbf{A}(\mathbf{c}-\mathbf{z}) + 2\mu \mathbf{A} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}] \mathbf{A}\mathbf{c} = \mathbf{0} \tag{55}$$

and hence:

$$\mathbf{c} + \mu [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}] \mathbf{A}\mathbf{c} = \mathbf{z} , \tag{56}$$

$$\mu \mathbf{B}\mathbf{A}\mathbf{c} = \mathbf{z} , \tag{57}$$

where:

$$\mathbf{B} = \mu^{-1}\mathbf{A}^{-1} + [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}} [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}] . \tag{58}$$

Calculation of the matrix $\mathbf{A}^{-1}$ is simple (since $\mathbf{A}$ is a diagonal matrix) and in calculating the matrix $\mathbf{B}^{-1}$ can be used specific properties of the matrix $\mathbf{B}$, including the fact that the matrix $\mathbf{E}$ appearing in (58) is a cyclic permutation matrix (a kind of circulant matrix), for which the decomposition can be performed to form a diagonal matrix with the use of Fourier matrix $\mathbf{W}$ [15]. The solution of equation (57) is (according to equation (A16) of Appendix A) the vector $\mathbf{c}$ defined as:

$$\mathbf{c} = \mathbf{z} - \frac{4\mu}{M} \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{\Lambda}_{\mathrm{A}}^2 & -j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} \\ j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} & \mathbf{\Lambda}_{\mathrm{B}}^2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{W}^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{\mathrm{H}} \end{bmatrix} \mathbf{A}\mathbf{z} \tag{59}$$

or by:

$$\mathbf{c} = \frac{1}{M} \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix} \cdot \left(\mathbf{I} - 4\mu \begin{bmatrix} \mathbf{\Lambda}_{\mathrm{A}}^2 & -j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} \\ j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} & \mathbf{\Lambda}_{\mathrm{B}}^2 \end{bmatrix} \mathbf{A} \right) \cdot \begin{bmatrix} \mathbf{W}^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{\mathrm{H}} \end{bmatrix} \mathbf{z} , \tag{60}$$

where: $\mathbf{W} = [e^{-j2\pi nk/M}]$ is a $M \times M$ matrix ($n, k = -M/2, ..., M/2-1$), and matrices $\mathbf{\Lambda}_{\mathrm{A}} = [\Lambda_{\mathrm{A}n}]$, $\mathbf{\Lambda}_{\mathrm{B}} = [\Lambda_{\mathrm{B}n}]$ are M-element diagonal matrices with elements defined by:

$$\Lambda_{\mathrm{A}n} = (\sin x_n) \cdot \left[1 + 12\mu R \left(\cos^2 x_n + \frac{1}{3}\sin^2 x_n \right) \right]^{-1/2} , \quad x_n = \frac{\pi n}{M} , \tag{61}$$

$$\Lambda_{\mathrm{B}n} = (\cos x_n) \cdot \left[1 + 12\mu R \left(\cos^2 x_n + \frac{1}{3}\sin^2 x_n \right) \right]^{-1/2} , \quad x_n = \frac{\pi n}{M} , \tag{62}$$

with the range $n = -M/2, ..., M/2-1$.

Using (27) in (60) and the fact that $\mathbf{W}^{\mathrm{H}}\mathbf{W}_R = M\cdot[\mathbf{0}\quad \mathbf{I}_{N\times N}\quad \mathbf{0}]^{\mathrm{T}}$ we have:

$$\mathbf{c} = \frac{1}{R}\begin{bmatrix}\mathbf{W}_R & 0\\ 0 & \mathbf{W}_R\end{bmatrix}\cdot\begin{bmatrix}\mathbf{I}-4\mu R(\mathbf{\Lambda}'_{\mathrm{A}})^2 & j12\mu R\cdot\mathbf{\Lambda}'_{\mathrm{A}}\mathbf{\Lambda}'_{\mathrm{B}}\\ -j4\mu R\cdot\mathbf{\Lambda}'_{\mathrm{A}}\mathbf{\Lambda}'_{\mathrm{B}} & \mathbf{I}-12\mu R(\mathbf{\Lambda}'_{\mathrm{B}})^2\end{bmatrix}\cdot\begin{bmatrix}\mathbf{w}'\\ \mathbf{w}''\end{bmatrix}, \tag{63}$$

where: $\mathbf{\Lambda}'_{\mathrm{A}} = [\Lambda_{\mathrm{A}n}]$, $\mathbf{\Lambda}'_{\mathrm{B}} = [\Lambda_{\mathrm{B}n}]$ are N-element diagonal matrices with elements defined by (61)-(62) for the range $n = -N/2,...,N/2-1$, and vectors $\mathbf{w}'$, $\mathbf{w}''$ are defined by (24)-(25) for the approximation from Section 3 and by (45)-(46) for the approximation from Section 4. Equation (63) can be transformed, using (61)-(62) and dependences for $\mathbf{w}'$, $\mathbf{w}''$, to the form:

$$\mathbf{c} = \frac{1}{R}\begin{bmatrix}\mathbf{W}_R & 0\\ 0 & \mathbf{W}_R\end{bmatrix}\cdot\begin{bmatrix}\mathbf{\Lambda}_\alpha\\ (1/6)\cdot\mathbf{\Lambda}_\beta\end{bmatrix}\cdot\mathbf{w},\quad \mathbf{c}' = \frac{1}{R}\mathbf{W}_R\mathbf{\Lambda}_\alpha\mathbf{w},\quad \mathbf{c}'' = \frac{1}{6R}\mathbf{W}_R\mathbf{\Lambda}_\beta\mathbf{w}, \tag{64}$$

where: $\mathbf{w}$ is the vector of the data window, diagonal matrices $\mathbf{\Lambda}_\alpha$, $\mathbf{\Lambda}_\beta$ are defined for the approximation from Section 3 as $\mathbf{\Lambda}_\alpha = [\alpha_n e^{-jx_n}]$, $\mathbf{\Lambda}_\beta = j[\beta_n e^{-jx_n}]$ and for the approximation from Section 4 as $\mathbf{\Lambda}_\alpha = [\alpha_n]$, $\mathbf{\Lambda}_\beta = j[\beta_n]$ with the coefficients α_n, β_n defined by:

$$\alpha_n(\mu) = (\mathrm{sinc}\ x_n)\frac{1+12\mu R\cos x_n \mathrm{sinc}\ x_n}{1+12\mu R(\cos^2 x_n + \frac{1}{3}\sin^2 x_n)},\quad \beta_n(\mu) = 6\frac{\mathrm{sinc}'\, x_n - 4\mu R\sin x_n \mathrm{sinc}^2 x_n}{1+12\mu R(\cos^2 x_n + \frac{1}{3}\sin^2 x_n)}. \tag{65}$$

Taking into account (64) and definitions $\mathbf{\Lambda}_\alpha$, $\mathbf{\Lambda}_\beta$, (35) has the form:

$$\mathbf{d} = \begin{bmatrix}\mathbf{W}_R & 0\\ 0 & \mathbf{W}_R\end{bmatrix}\cdot\begin{bmatrix}\mathbf{\Lambda}_\alpha\\ \mathbf{\Lambda}_\beta\end{bmatrix}\cdot\mathbf{w},\quad \mathbf{d}' = \mathbf{W}_R\mathbf{\Lambda}_\alpha\mathbf{w},\quad \mathbf{d}'' = \mathbf{W}_R\mathbf{\Lambda}_\beta\mathbf{w} \tag{66}$$

or in the non-matrix form, from (33)-(34) for the approximation from Section 3:

$$d'_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) = a_k\frac{k+1/2}{R} + b_k = \sum_{n=-N/2}^{N/2-1} w_n\alpha_n(\mu)e^{-jx_n}e^{-j2\pi nk/M}, \tag{67}$$

$$d''_k = \hat{W}_k\left(\frac{k+1}{R}\right) - \hat{W}_k\left(\frac{k}{R}\right) = \frac{a_k}{R} = j\sum_{n=-N/2}^{N/2-1} w_n\beta_n(\mu)e^{-jx_n}e^{-j2\pi nk/M} \tag{68}$$

and similarly, from (47)-(48) for the approximation from Section 4:

$$d'_k = \hat{W}_k\left(\frac{k}{R}\right) = a_k\frac{k}{R} + b_k = \sum_{n=-N/2}^{N/2-1} w_n\alpha_n(\mu)e^{-j2\pi nk/M}, \tag{69}$$

$$d''_k = \hat{W}_k\left(\frac{k+1/2}{R}\right) - \hat{W}_k\left(\frac{k-1/2}{R}\right) = \frac{a_k}{R} = j\sum_{n=-N/2}^{N/2-1} w_n\beta_n(\mu)e^{-j2\pi nk/M}. \tag{70}$$

For $\mu = 0$ the Eqs. (67)-(68) are identical with (37)-(38), and (69)-(70) are identical with (49)-(50).

Let us notice that for $\mu = 0$ (64) is the same as (27), i.e. $\mathbf{z} = \mathbf{c}(\mu = 0)$. From this fact and from (64) the expression $(\mathbf{c}-\mathbf{z})^{\mathrm{H}}\mathbf{A}(\mathbf{c}-\mathbf{z})$ is determined. Using additional relations (30)-(31) the total mean square error Q of approximation defined by (7)-(8) or (39) is obtained:

$$Q(\mu, R) = N\sum_{n=-N/2}^{N/2-1} c_n(\mu)\cdot w_n^2, \tag{71}$$

where:

$$c_n(\mu, R) = 1 - \alpha_n^2(0) - \frac{1}{12}\beta_n^2(0) + [\alpha_n(\mu) - \alpha_n(0)]^2 + \frac{1}{12}[\beta_n(\mu) - \beta_n(0)]^2 . \tag{72}$$

For $\mu = 0$ the Eqs (71)-(72) are the same as (32).

The Maclaurin series of the $c_n(\mu)$ for $\mu = 0$ and $\mu \to \infty$ are described by:

$$c_n(\mu = 0) = \frac{x_n^4}{45} - \frac{4x_n^6}{1575} + \frac{2x_n^8}{14175} - \frac{16x_n^{10}}{3274425} + o(x_n^{12}) , \tag{73}$$

$$c_n(\mu \to \infty) = \frac{x_n^4}{45} + \frac{4x_n^6}{189} + \frac{14x_n^8}{2025} + \frac{16x_n^{10}}{18711} + o(x_n^{12}) . \tag{74}$$

These series and (20) show that for high values of R, the $Q(\mu \to \infty) \to Q(\mu = 0)$, what is also confirmed by Fig. 2 and the figures presented in Section 6.

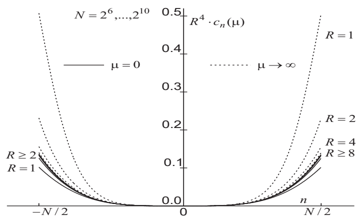

Fig. 2. Values of $R^4 c_n(\mu)$ versus n as obtained from (72)

6. Exemplary approximations for triangle window

The conclusions obtained at the end of Section 5 and in Fig. 2 are confirmed for the case of calculation of (71) for the triangle window – the mean square error $Q(\mu, R)$ is proportional to R^{-4}: when R is doubled, the value $c_n(\mu)$ to decrease 16-fold (Fig. 2 and (71)-(72)), which results in the 16-fold reduction of the mean square error $Q(\mu, R)$, as is shown in Fig. 3a,b, especially for big R. The value of the $Q(\mu, R)$ depends also on the value of parameter μ – for small R this dependence is significant, but the higher the value R, the dependence on μ is lesser (Fig. 2, 3a,b). The influence of the value μ is clearly shown in Fig. 3c in circle zooms – for $\mu = 0$ the discontinuity of the approximation is the largest at analyzed point $\lambda = 0.75$ bins (although the mean square error Q has a minimum, as shown in Fig. 3a,b), for $\mu = 0.1$ this discontinuity is more then twice smaller (but the Q is increased), and for $\mu \to \infty$ approximation is continuous (but the error Q is significantly larger for small R, as shown in Fig. 3b). The cases for the triangle window, $\mu = 0$ and $R = 2$ are also shown in Fig. 1.

For the data window, the most important feature is its frequency characteristics $|W(\lambda)|$. This characteristic is completed by a graph of the frequency characteristic of the approximation error $|W(\lambda) - \hat{W}(\lambda)|$ (Fig. 4). For $R \geq 2$ and triangle window, the envelope of this error is about $(2.4R^2)$-times smaller than the envelope of the triangle window leakage (Fig. 4). The approximation error $|W(\lambda) - \hat{W}(\lambda)|$ is discontinuous for $\mu = 0$ and continuous for $\mu \to \infty$. The differences in Fig. 4 are the most visible for small value of R, i.e. $R = 2$.

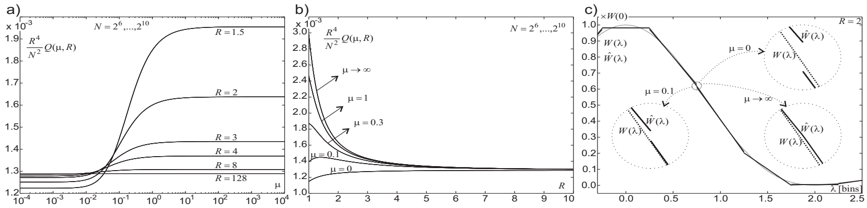

Fig. 3. The quality of approximation (example for a triangle window): a,b) dependence of the mean square error Q from μ and R in a graph $R^4N^{-2}Q$ versus μ and R, c) different degree of the approximation discontinuity at a point $\lambda = 0.75$ bins magnified in three circle zooms for $\mu = 0,\ 0.1,\ \infty$

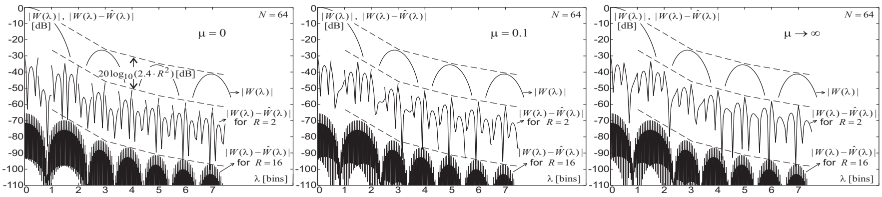

Fig. 4. Graph of $|W(\lambda)|$ and $|W(\lambda)-\hat{W}(\lambda)|$ for a triangle window and $\mu = 0,\ 0.1,\ \infty$

7. Conclusions

The main result of the paper are the Eqs. (65)-(70) for the parameters d'_k, d''_k of the approximating linear functions (and hence their coefficients a_k, b_k) for the given value of the parameter μ, which defines the degree of the approximation discontinuity. The special cases are obtained for $\mu = 0$ (largest discontinuity, smallest mean square error of approximation) and $\mu \to \infty$ (continuity approximation, largest mean square error). The values of coefficients of approximating linear functions are easy to calculate by FFT algorithm with the use of zero padding technique. The obtained formulas are valid for both kinds of approximation: when approximating linear functions are defined for the range $[k/R,(k+1)/R]$ bins (Fig. 1a,b), and for the range $[(k-1/2)/R,(k+1/2)/R]$ bins (Fig. 1c,d). Results obtained in the paper are valid for any data window, not only symmetric like the triangle window used as an example, because all formulas are obtained without any assumption of this type. The formulas for parameters of approximating functions are given for the shifted DtFT, but it is possible to use them also for the non-shifted DtFT, by simple multiplication by the function $\exp(-j\pi k/R)$, as follows from (2)-(4).

Appendix A. Selected mathematical equations

$$\int_{k/R}^{(k+1)/R} d\lambda = \frac{1}{R}, \qquad \int_{k/R}^{(k+1)/R} \lambda d\lambda = \frac{k+1/2}{R^2}, \qquad \int_{k/R}^{(k+1)/R} \lambda^2 d\lambda = \frac{(k+1/2)^2}{R^3} + \frac{1}{12R^3}, \tag{A1}$$

$$\int_{(k-1/2)/R}^{(k+1/2)/R} d\lambda = \frac{1}{R}, \qquad \int_{(k-1/2)/R}^{(k+1/2)/R} \lambda d\lambda = \frac{k}{R^2}, \qquad \int_{(k-1/2)/R}^{(k+1/2)/R} \lambda^2 d\lambda = \frac{k^2}{R^3} + \frac{1}{12R^3}, \tag{A2}$$

$$\begin{aligned} \left|W(\lambda)-\hat{W}_k(\lambda)\right|^2 &= [W(\lambda)-a_k\lambda-b_k][W^*(\lambda)-a_k^*\lambda-b_k^*] \\ &= |W(\lambda)|^2 + [(a_k^*\lambda+b_k^*)(a_k\lambda+b_k)] - [a_k^*\lambda W(\lambda)+b_k^* W(\lambda)] - [a_k\lambda W^*(\lambda)+b_k W^*(\lambda)] \end{aligned}. \tag{A3}$$

Based on (2), (9) there is:

$$\int_{-N/2}^{N/2} |W(\lambda)|^2 \, d\lambda = N \sum_{n=-N/2}^{N/2-1} w_n^2 = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w}\,. \tag{A4}$$

Based on (A1), (10)-(13) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{k/R}^{(k+1)/R} (a_k^* \lambda + b_k^*)(a_k \lambda + b_k) d\lambda = \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{c}\,. \tag{A5}$$

Based on (10)-(16) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{k/R}^{(k+1)/R} (a_k \lambda + b_k)^* W(\lambda) d\lambda = \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{z}\,. \tag{A6}$$

Based on (A3)-(A6) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{k/R}^{(k+1)/R} \left| W(\lambda) - \hat{W}_k(\lambda) \right|^2 d\lambda = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w} + \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{c} - \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{z} - \mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{c}\,. \tag{A7}$$

Based on (A2), (10)-(11), (13), (40) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{(k-1/2)/R}^{(k+1/2)/R} (a_k^* \lambda + b_k^*)(a_k \lambda + b_k) d\lambda = \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{c}\,. \tag{A8}$$

Based on (10)-(11), (13)-(14), (40)-(42) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{(k-1/2)/R}^{(k+1/2)/R} (a_k \lambda + b_k)^* W(\lambda) d\lambda = \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{z}\,. \tag{A9}$$

Based on (A3)-(A4), (A8)-(A9) there is:

$$\sum_{k=-M/2}^{M/2-1} \int_{(k-1/2)/R}^{(k+1/2)/R} \left| W(\lambda) - \hat{W}_k(\lambda) \right|^2 d\lambda = N \cdot \mathbf{w}^{\mathrm{T}}\mathbf{w} + \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{c} - \mathbf{c}^{\mathrm{H}}\mathbf{A}\mathbf{z} - \mathbf{z}^{\mathrm{H}}\mathbf{A}\mathbf{c}\,. \tag{A10}$$

For any $\mathbf{x} = [x_{-M/2}, ..., x_{M/2-1}]^{\mathrm{T}}$, $\mathbf{y} = [y_{-M/2}, ..., y_{M/2-1}]^{\mathrm{T}}$ meeting the condition $x_{k+M} = x_k$, the matrix $\mathbf{E}$ from (54) and identity matrix $\mathbf{I}$, there is:

$$\sum_{k=-M/2}^{M/2-1} x_{k+1} y_k^* = \mathbf{y}^{\mathrm{H}}\mathbf{E}\mathbf{x}\,, \qquad \sum_{k=-M/2}^{M/2-1} x_{k+1}^* y_k = \mathbf{x}^{\mathrm{H}}\mathbf{E}^{\mathrm{T}}\mathbf{y}\,. \tag{A11}$$

For any $\mathbf{x} = [x_{-M/2}, ..., x_{M/2-1}]^{\mathrm{T}}$, $\mathbf{y} = [y_{-M/2}, ..., y_{M/2-1}]^{\mathrm{T}}$ meeting the conditions $x_{k+M} = x_k$, $y_{k+M} = y_k$, $\mathbf{z} = [\mathbf{x}^{\mathrm{T}} \quad \mathbf{y}^{\mathrm{T}}]^{\mathrm{T}}$, the matrix $\mathbf{E}$ from (54) and identity matrix $\mathbf{I}$, based on (A11), there is:

$$\sum_{k=-M/2}^{M/2-1} \left| (x_{k+1} - x_k) - (y_{k+1} + y_k) \right|^2 = \mathbf{z}^{\mathrm{H}}[\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}}[\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]\mathbf{z}\,. \tag{A12}$$

For diagonal matrices $\mathbf{\Lambda}_i$ ($i = 1, 2, 3$), orthogonal matrix $\mathbf{W}$ meeting the condition $\mathbf{W}\mathbf{W}^{\mathrm{H}} = \mathbf{W}^{\mathrm{H}}\mathbf{W} = M \cdot \mathbf{I}$ there is for the following block matrices:

$$\begin{bmatrix} \mathbf{W}\mathbf{\Lambda}_1\mathbf{W}^{\mathrm{H}} & \mathbf{W}\mathbf{\Lambda}_3\mathbf{W}^{\mathrm{H}} \\ -\mathbf{W}\mathbf{\Lambda}_3\mathbf{W}^{\mathrm{H}} & \mathbf{W}\mathbf{\Lambda}_2\mathbf{W}^{\mathrm{H}} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{W}\mathbf{\Lambda}_2(\mathbf{\Lambda}_1\mathbf{\Lambda}_2+\mathbf{\Lambda}_3^2)^{-1}\mathbf{W}^{\mathrm{H}} & -\mathbf{W}\mathbf{\Lambda}_3(\mathbf{\Lambda}_1\mathbf{\Lambda}_2+\mathbf{\Lambda}_3^2)^{-1}\mathbf{W}^{\mathrm{H}} \\ \mathbf{W}\mathbf{\Lambda}_3(\mathbf{\Lambda}_1\mathbf{\Lambda}_2+\mathbf{\Lambda}_3^2)^{-1}\mathbf{W}^{\mathrm{H}} & \mathbf{W}\mathbf{\Lambda}_1(\mathbf{\Lambda}_1\mathbf{\Lambda}_2+\mathbf{\Lambda}_3^2)^{-1}\mathbf{W}^{\mathrm{H}} \end{bmatrix} = M^2\mathbf{I}\,. \tag{A13}$$

For the matrix $\mathbf{W} = [e^{-j2\pi nk/M}]$ ($n, k = -M/2, ..., M/2-1$), matrix $\mathbf{E}$ from (54) and diagonal matrix $\mathbf{\Lambda}_e = [e^{-j2\pi n/M}]$ there is:

$$M \cdot \mathbf{E} = \mathbf{W}\mathbf{\Lambda}_e\mathbf{W}^{\mathrm{H}}\,, \qquad M \cdot \mathbf{E}^{\mathrm{T}} = \mathbf{W}\mathbf{\Lambda}_e^*\mathbf{W}^{\mathrm{H}}\,, \qquad \mathbf{\Lambda}_e = [e^{-j2\pi n/M}]\,. \tag{A14}$$

For the matrices: $\mathbf{E}$ from (A14), $\mathbf{A}$ from (10) and identity matrix $\mathbf{I}$ there is:

$$\mathbf{B} = \mu^{-1}\mathbf{A}^{-1} + [\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]^{\mathrm{T}}[\mathbf{I}-\mathbf{E} \quad \mathbf{I}+\mathbf{E}]$$
$$= \frac{1}{M}\begin{bmatrix} \mathbf{W}[(2+(\mu R)^{-1})\mathbf{I}-(\mathbf{\Lambda}_e+\mathbf{\Lambda}_e^*)]\mathbf{W}^{\mathrm{H}} & \mathbf{W}(\mathbf{\Lambda}_e-\mathbf{\Lambda}_e^*)\mathbf{W}^{\mathrm{H}} \\ -\mathbf{W}(\mathbf{\Lambda}_e-\mathbf{\Lambda}_e^*)\mathbf{W}^{\mathrm{H}} & \mathbf{W}[(2+(3\mu R)^{-1})\mathbf{I}+(\mathbf{\Lambda}_e+\mathbf{\Lambda}_e^*)]\mathbf{W}^{\mathrm{H}} \end{bmatrix}. \tag{A15}$$

For $\mathbf{\Lambda}_1 = (2+(\mu R)^{-1})\mathbf{I}-(\mathbf{\Lambda}_e+\mathbf{\Lambda}_e^*)$, $\mathbf{\Lambda}_2 = (2+(3\mu R)^{-1})\mathbf{I}+(\mathbf{\Lambda}_e+\mathbf{\Lambda}_e^*)$, $\mathbf{\Lambda}_3 = \mathbf{\Lambda}_e - \mathbf{\Lambda}_e^*$ in (A15) there is from (A13):

$$\mathbf{B}^{-1} = \frac{\mu}{M}\begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix} \cdot \left(\mathbf{I} - 4\mu\mathbf{A}\begin{bmatrix} \mathbf{\Lambda}_{\mathrm{A}}^2 & -j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} \\ j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} & \mathbf{\Lambda}_{\mathrm{B}}^2 \end{bmatrix} \right) \cdot \mathbf{A}\begin{bmatrix} \mathbf{W}^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{\mathrm{H}} \end{bmatrix}$$
$$= \mu\mathbf{A} - \frac{4\mu^2}{M}\mathbf{A}\begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W} \end{bmatrix}\begin{bmatrix} \mathbf{\Lambda}_{\mathrm{A}}^2 & -j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} \\ j\mathbf{\Lambda}_{\mathrm{A}}\mathbf{\Lambda}_{\mathrm{B}} & \mathbf{\Lambda}_{\mathrm{B}}^2 \end{bmatrix}\begin{bmatrix} \mathbf{W}^{\mathrm{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{\mathrm{H}} \end{bmatrix}\mathbf{A}, \tag{A16}$$

where: $\mathbf{\Lambda}_{\mathrm{A}}$, $\mathbf{\Lambda}_{\mathrm{B}}$ are defined by (61)-(62).

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