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ALGORITHMIC METHOD FOR PHASE ANGLE SHIFT OF NOISY VOLTAGES USING CONDITIONAL AVERAGING OF DELAYED SIGNAL'S ABSOLUTE VALUE

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Abstract

The method of a phase shift angle measurement using conditional averaging of delayed signal absolute value (CAAV) is presented in this paper. The input sinusoidal signal $x(t)$ is without noise. White noise with normal distribution and band limited to low frequencies has been applied as disturbance of delayed sinusoidal signal $z(t)$. Noise $n(t)$ – $N(0, \sigma_n)$ is added to the delayed signal - the noised and delayed signal $z(t)$ is obtained. The phase angle shift is proportional to time location of CAAV's minimum (minimum of the characteristic of conditional averaging of delayed signal's absolute value). The phase angle shift can be determined on the basis of conditional averaging value of elaborated algorithm. The characteristics of conditional average of delayed signal's absolute value in the surrounding of the minimum of this function (the results of practical investigations and theoretical calculation) are presented. The experimental variance of characteristic CAAV in surroundings of the minimum (obtained from practical investigations and calculation) is illustrated in the paper. The algorithms of conditional averaging have been elaborated and practically realized in the LabVIEW environment.

Keywords: phase angle shift, additive disturbance, conditional signal averaging.

1. Introduction

The most common disturbances encountered in the measurements of the phase angle shift of two sinusoidal signals result from the noise, harmonics and constants of the signal. Electronic phase-meters with the processing of phase angle shift into time intervals are susceptible to noise. Random disturbances affect the precision of zero passage of both runs and directly affect the measurement precision of the time segment which corresponds to the phase shift φ of two analysed signals [1]. In real-world measurements two cases are observed: both signals (input and output) are noised or only the delayed signal $z(t)$ is noised.

The models with random disturbances occur in the evaluation of the measurement accuracy of small angle for example in optical interferometers [2]. A correction of the measurement accuracy of the phase shift angle of noisy signals can be obtained by using algorithm methods including statistical analysis, e.g. determination of the cross correlation of two signals shifted by angle φ [1, 3, 4], as well as algorithms which use conditional averaging of signals as proposed by the authors of the paper [5, 6, 7, 8].

2. Models of the estimated characteristics

The signal processing model assumes that the original sinusoidal signal $x(t)$ is free from disturbances, while the signal additively disturbed with noise $n(t)$, characterized by the model $N(0, \sigma_n)$, is the secondary signal $y(t)$. The signal available for analysis is the signal $z(t) = y(t) + n(t)$. The signal values occur in the moments t_1 and t_2 ($\tau = t_2 - t_1$).

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The description of the signal processing is made with the following denotation:

$$x_1 = x(t_1) = X_m \cos(\omega \cdot t_1 + \varphi_x), \tag{1}$$

$$y_1 = y(t_1) = Y_m \cos(\omega \cdot t_1 + \varphi_y), \tag{2}$$

$$y_2 = y(t_1 + \tau) = y(t_2) = Y_m \cos(\omega \cdot t_1 + \omega \cdot \tau + \varphi_{xy}), \tag{3}$$

where: $\varphi_{xy} = \varphi_x - \varphi_y$ is the phase shift angle between signals $x(t)$ and $y(t)$,

$$n_2 = n(t_2), \tag{4}$$

$$z_2 = z(t_2) = y(t_2) + n(t_2). \tag{5}$$

The following relation is right for constant value of phase angle shift between signals $x(t)$ and $y(t)$:

$$y_2 = Y_m \cos\left(\omega \cdot \tau + \varphi_{xy} \pm arc\cos \frac{x_1}{X_m}\right). \tag{6}$$

The relation is unique in the set 0 - π and equal to:

$$y_2 = Y_m \cos\left(\omega \cdot \tau + \varphi_{xy} + arc\cos \frac{x_1}{X_m}\right). \tag{7}$$

The conditional probability density for the functional relation (7) is equal to:

$$p\left(y_2\big|_{x_1}\right) = p(y_{2w}) = \delta\left[y_2 - Y_m \cos\left(\omega \cdot \tau + \varphi_{xy} + arc\cos \frac{x_1}{X_m}\right)\right]. \tag{8}$$

The conditional probability density for the independent signals $n(t)$ and $x(t)$ is:

$$p\left(n_2\big|_{x_1}\right) = p(n_{2w}) = p(n_2) = \frac{1}{\sigma_n \sqrt{2\pi}} e^{-\frac{n^2}{2\sigma_n^2}}. \tag{9}$$

In such a case the conditional density of signal $y(t)$ as phase-shifted and disturbed by noise $n(t)$ can be determined with a conditional density convolution of the constituent signals:

$$\begin{aligned} \mathrm{p}\left(z_2\big|_{x_1}\right) &= \mathrm{p}(z_{2w}) = \int_{\infty}^{-\infty} \delta\left[y_2 - Y_m \cos\left(\omega \cdot \tau + \phi_{xy} + arccos\frac{x_1}{X_m}\right)\right] \cdot \frac{1}{\sigma_n\sqrt{2\pi}} e^{-\frac{(z_{2w}-y_2)^2}{2\sigma_n^2}} dy_2 = \\ &= \frac{1}{\sigma_n\sqrt{2\pi}} e^{\frac{\left[z_{2w} - Y_m \cos\left(\omega\cdot\tau + \phi_{xy} + arccos\frac{x_1}{X_m}\right)\right]^2}{2\sigma_n^2}} \end{aligned}. \tag{10}$$

When the threshold x_1=0, the following occurs:

$$p\left(z_2\big|_{x_1=0}\right) = p(z_{20}) = \frac{1}{\sigma_n \sqrt{2\pi}} e^{-\frac{[z_{20}+y_0]^2}{2\sigma_n^2}}, \tag{11}$$

where: $y_0 = -\ Y_m \sin(\omega \cdot \tau + \varphi_{xy})$.

In order to analyze the non-linear transformation of the signal z_{20} the following denotations are introduced:

$$w_2 = w_2\big|_{x_1=0} = \left|z_2\big|_{x_1=0}\right| = |z_{20}|\,;\, z_{201} = +w_2\,;\, z_{202} = -w_2. \tag{12}$$

The probability density of the absolute value of conditional signal z_{20} can be expressed by formula:

$$p_{w2}(w_2)=\frac{p_{z2}(z_{201})}{\left|\frac{dw_2}{dz_2}\right|z_{201}}+\frac{p_{z2}(z_{202})}{\left|\frac{dw_2}{dz_2}\right|z_{202}}=\frac{1}{\sigma_n\sqrt{2\pi}}\left[e^{-\frac{(w_2-y_0)^2}{2\sigma_n^2}}+e^{-\frac{(w_2+y_0)^2}{2\sigma_n^2}}\right]. \tag{13}$$

The conditional average value of the absolute value of signal z_{20} (CAAV) can be derived from the following expression:

$$E\left[\|z_2\|_{x_1=0}\right]=\overline{w}_2=\int_0^\infty w_2 p_{w2}(w_2)\,dw_2=\frac{1}{\sigma_n\sqrt{2\pi}}\int_0^\infty w_2 e^{-\frac{(w_2-y_0)^2}{2\sigma_n^2}}dw_2+\frac{1}{\sigma_n\sqrt{2\pi}}\int_0^\infty w_2 e^{-\frac{(w_2+y_0)^2}{2\sigma_n^2}}dw_2=I_1+I_2\,. \tag{14}$$

The first integral, I_1, can be calculated by introducing an auxiliary variable:

$$\frac{w_2-y_0}{\sigma_n}=\eta\,;\ \eta_0=-\frac{y_0}{\sigma_n}\,. \tag{15}$$

The calculations give the following result:

$$I_1=y_0\left[\frac{1}{2}+\Phi(\eta_0)\right]+\frac{\sigma_n}{\sqrt{2\pi}}e^{-\frac{y_0^2}{2\sigma_n^2}}, \tag{16}$$

where: $\Phi(\eta_0)=\frac{1}{\sqrt{2\pi}}\int_0^{\eta_0}e^{-\frac{\eta^2}{2}}d\eta$ is the Laplace function.

The integral I_2 can be calculated in a similar way:

$$I_2=-y_0\left[\frac{1}{2}-\Phi(\eta_0)\right]+\frac{\sigma_n}{\sqrt{2\pi}}e^{-\frac{y_0^2}{2\sigma_n^2}}\,. \tag{17}$$

When substituting with (16) and (17) to (14), the following expression for the CAAV is obtained:

$$\overline{w}_2=\overline{w}_2(\tau)=2y_0\cdot\Phi(\eta_0)+\sqrt{\frac{2}{\pi}}\sigma_n\cdot e^{-\frac{y_0^2}{2\sigma_n^2}}\,. \tag{18}$$

The function $\Phi(\eta_0)$ can be represented by a series:

$$\Phi(\eta_0)=\frac{1}{\sqrt{\pi}}e^{-\frac{\eta^2}{2}}T_k\sum_{k=0}^{\infty}\left(\frac{\eta_0}{\sqrt{2}}\right)^{2k+1}, \tag{19}$$

where: $T_k=\frac{2^k}{1\cdot 3\cdots(2k+1)}$.

When the values η_0 are low and the only first term of the series is used, the model of linear approximation is obtained:

$$\Phi(\eta_0)\approx\frac{\eta_0}{\sqrt{2\cdot\pi}}\,. \tag{20}$$

Following further reductive transformations of the expression (18) using the formulas (11) and (15), the result is the relation for the conditional average value in the direct neighbourhood of its minimum:

$$\overline{w}_2(\tau) \approx \sqrt{\frac{2}{\pi}}\sigma_n \left[\left(\frac{Y_m}{\sigma_n} \right)^2 sin^2(\omega\tau + \varphi_{xy}) + 1 \right]. \tag{21}$$

At the minimum point of the characteristic $\overline{w}_2(\tau)$ for $\eta_0 = 0$:

$$\overline{w}_2 = \sqrt{\frac{2}{\pi}}\sigma_n, \tag{22}$$

$$p(w_2) = \sqrt{\frac{2}{\pi}}\frac{1}{\sigma_n} e^{-\frac{w_2^2}{2\sigma_n^2}}, \tag{23}$$

$$Var[w_2] = \int_0^\infty (w_2 - \overline{w_2})^2 \, p(w_2)\, dw_2 = \sigma_n^2\left(1 - \frac{2}{\pi}\right) \approx 0{,}36\sigma_n^2. \tag{24}$$

At the minimum point $\overline{w}_2(\tau)$ for the delay τ_0 , the variance of the variable w_2 is 36% of the variance of the sinusoidal signal which transits through the zero value and is additively disturbed by noise $N(0,\sigma_n)$.

At the characteristic points $\overline{w}_2(\tau)$, which are significantly distant from the minimum point $\overline{w}_2(\tau_0)$ for $\eta_0 \geq 3$, based on (18):

$$\overline{w}_2(\tau) = \overline{w}_2 \approx |y_0|. \tag{25}$$

The mean-square value of the variable w_2 can be expressed with the following relation:

$$\psi_{w_2}^2 = E\left\{ \left| \, |y_2|_{x_1} + n_2 \right|^2 \right\} = E\left\{ (y_0 + n)^2 \right\} = E(y_0^2) + 2y_0 E(n) + E(n^2) = y_0^2 + \sigma_n^2, \tag{26}$$

while the variance is:

$$Var[w_2] = \psi_{w_2}^2 - \overline{w}_2^2 = \sigma_n^2. \tag{27}$$

The variance of the variable w_2 is the disturbance variance.

When the values of $\pm\Delta\tau$ are low in the neighbourhood of the point τ_0 which determines the minimum of the function $\overline{w}_2(\tau)$, it is possible to further reduce the relation (21) to the following:

$$\hat{\overline{w}}_2 \approx \overline{w}_2(\tau_0)\left[\left(\frac{Y_m}{\sigma_n}\omega\Delta\tau \right)^2 + 1 \right]. \tag{28}$$

The expected value is:

$$E\left[\hat{\overline{w}}_2(\tau_0)\right] \approx \overline{w}_2(\tau_0) \quad . \tag{29}$$

The characteristic variance is:

$$Var\left[\hat{\overline{w}}_2(\tau_0)\right] = E\left\{\left[\hat{\overline{w}}_2(\tau_0) - \overline{w}_2(\tau_0)\right]^2\right\} = \overline{w}_2{}^2(\tau_0)\left(\frac{Y_m}{\sigma_n}\omega\right)^4 E\left(\Delta\tau^4\right). \tag{30}$$

With the distribution model for the deviations $\Delta\tau$ from τ_0 in the form of $N(0,\sigma_{\Delta t})$ the following is obtained:

$$E\left(\Delta\tau^4\right) = 3\sigma_{\Delta\tau}^4. \tag{31}$$

The relative uncertainty square is:

$$u_{rel}^2\left[\hat{\overline{w}}_2(\tau_0)\right] = \frac{Var\left[\hat{\overline{w}}_2(\tau_0)\right]}{\left[\overline{w}_2(\tau_0)\right]^2} = 3\left(\frac{Y_m\omega}{\sigma_n}\right)^4 \sigma_{\Delta\tau}^4. \tag{32}$$

This allows one to calculate the deviation $\sigma_{\Delta\tau}$ from the following relation:

$$\sigma_{\Delta\tau} = 0{,}76\frac{\sigma_n}{Y_m \cdot \omega}\left\{u_{rel}\left[\hat{\overline{w}}_2(\tau_0)\right]\right\}^{\frac{1}{2}}. \tag{33}$$

The resulting calculations (22) and (24) also imply that:

$$u_{rel}^2\left[\hat{\overline{w}}_2(\tau_0)\right] = \frac{Var\left[\hat{\overline{w}}_2(\tau_0)\right]}{\left[\overline{w}_2(\tau_0)\right]^2} = \frac{\sigma_n^2\left(1 - \frac{2}{\pi}\right)}{\left(\sqrt{\frac{2}{\pi}}\sigma_n\right)^2} = 0{,}57 \cdot \tag{34}$$

Taking the expressions (33) and (34) into account gives an approximate theoretical relation for the standard deviation of the estimate $\hat{\tau}_0$:

$$\sigma_{\hat{\tau}_0} = \sigma_{\Delta\tau} \approx 0{,}66\frac{\sigma_n}{Y_m \cdot \omega}. \tag{35}$$

It can be proven that the standard deviation of the moment of the sinusoidal signal transiting through the zero level (2) and additively disturbed by the noise $N(0,\sigma_n)$ is formulated as follows:

$$\sigma_{\tau_0} \approx \frac{\sigma_n}{Y_m \cdot \omega}. \tag{36}$$

The comparison of the expressions (35) and (36) shows a decrease in the standard deviation of the CAAV minimum location by approximately 34%.

The experimental characteristic of the CAAV is obtained by conditional averaging of delayed signal $z(t)$:

$$\hat{\overline{w}}_2(\tau) = \frac{1}{M}\sum_{i=1}^{M} w_{2i}\left(t_i + \tau\right)\Big|_{x(t_i)=0}, \tag{37}$$

with the experimental variance for the argument τ_0 based on (24) and equal to:

$$Var\left[\hat{\overline{w}}_2\right] = \frac{0{,}36\sigma_n^2}{K}, \tag{38}$$

where: K is the number of the CAAV values calculated in this point. As for the characteristic points which are significantly distant from the minimum point (for $\eta_0 \geq 3$):

$$Var\left[\hat{\overline{w}}_2\right] = \frac{\sigma_n^2}{K}. \tag{39}$$

The argument of the minimum of the CAAV characteristic (τ_0), allows determination of the sought phase shift from the following relation:

$$\hat{\varphi}_{xy} = -\,\hat{\tau}_0 \cdot \omega. \tag{40}$$

with the standard uncertainty of:

$$\sigma_{\hat{\varphi}_{xy}} = \sigma_{\hat{\tau}_0} \cdot \omega = 0{,}66 \cdot \frac{\sigma_n}{Y_m}. \tag{41}$$

3. Practical analysis

The measurements were performed using a laboratory stand equipped with a digital generator for sinusoidal voltage signals with input phase shift, random voltage signal generators, models of disturbing systems, a data acquisition system with statistical analysis software and a digital oscilloscope. The measurements included the analysis of the following signals: $x(t)$ (disturbance-free) with the amplitude $X_m = 1$ V and the frequency f_x=100 Hz; and the signal $z(t)$ delayed by 1.046(6) rad (and disturbed at the $SNR = X_m^2/2\sigma_n^2 = 294$). The signals $x(t)$ and $z(t)$ were sampled with the frequency f_p=240 kHz [9].

Figure 1 presents the runs of the conditional average value characteristics of the delayed and noisy signal absolute value (CAAV): both practical and approximated based on the expression (28). Both characteristics are consistent in the direct neighborhood of the minimum points.

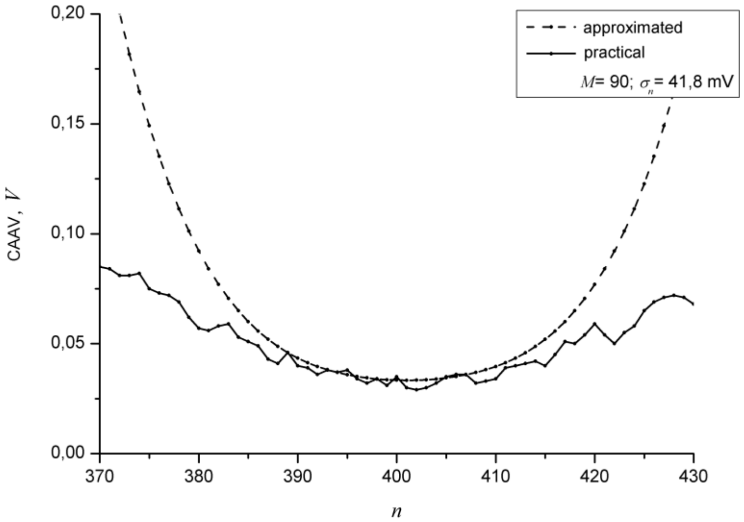

Fig. 1. The characteristics of conditional averaging of absolute value of the delayed signal surrounding its minimum.

The relation between the experimental (practical) variance of the CAAV characteristics in the minimum neighborhood is presented in Fig. 2. The calculations were based on M = 90 of the performance of the conditional averaging of the signal $z(t)$ and K =100 of the CAAV characteristic calculations at every point. The sinusoidal signals had the amplitude $X_m = Y_m$ =1 V, while the standard deviation of the noise was σ_n = 0,0418 V. The consistency of the practical characteristic with the theoretical calculations of values can be observed:

- In the direct neighborhood of the minimum – expression (38) (lower broken line);
- At the points significantly distant from the minimum – expression (39) (upper broken line).

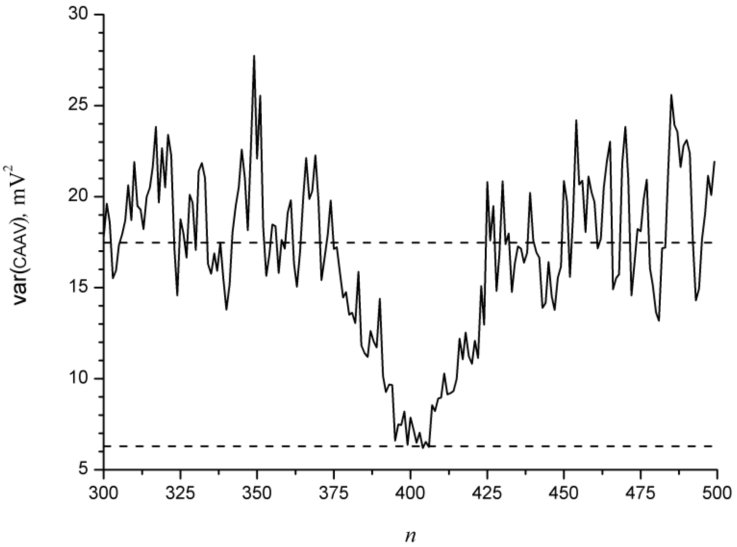

Fig. 2. The experimental variance of characteristic CAAV close to the minimum.

4. Conclusion

The application of conditional averaging of the absolute value of the delayed sinusoidal signal disturbed by normal noise allows the variance to be reduced by the characteristic delay time τ_0 proportional to the phase shift between the signals in relation to the zero-transition moment variance of the sinusoidal signal on the same disturbance – using a traditional measurement method.

The precision of the estimate $\hat{\tau}_0$ based on the CAAV characteristic minimum is determined by the form (rate of rise) and variance of the characteristic itself. The study shows the consistency of the theoretical and practical parameters of the CAAV characteristic at the minima.

The decrease in the variance $\sigma^2_{\hat{\tau}_0}$ by 56 % may have practical use in the measurements of voltage phase shifts at infra-low frequencies (for $f < 1$ Hz) due to the benefit of shortening the length of the time for the analysis at the assumed measurement precision.

Further improvement of the precision of the estimated value $\hat{\tau}_0$ proportional to the measured phase shift can be obtained by using a modified method of multi-level conditional averaging of the delayed signal absolute value.

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