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ON A CERTAIN CLASS OF EXPANDING SYSTEMS

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Abstract

The interesting properties of a class of expanding systems are discussed. The operation of the considered systems can be described as follows: the input signal is processed by a linear dynamic converter in subsequent time intervals, each of them is equal to T_i. Processing starts at the moments $n \cdot T_i$, always after zeroing of converter initial conditions. For smooth input signals and a given transfer function of the converter one can suitably choose T_i and the gain coefficient in order to realize the postulated linear operations on input signals, which is quite different comparing it to the operation realized by the converter. The errors of postulated operations are mainly caused by non-smooth components of the input signal. The principles for choice of system parameters and rules for system optimization are presented in the paper. The referring examples are attached too.

Keywords: Expanding Systems, conditions and accuracy of action.

1. Introduction

The idea of so called "expanding systems" [1] was presented fifty years ago. This idea has been used for several applications in metrology (dot recorders, Keinath's compensator, nonlinear static converters for analog computers and other). The current paper deals with a certain class of expanding systems realizing dynamical operations on input signals. According to classification rules given by F.E. Tiemnikov [1] one can treat them as "passive expanding systems". The systems of this type can be assigned for dynamical operations on "smooth" input signals. The easily feasible converters with suitably tuned parameters (like oscillatory ones) can be used as units processing the input signals. The expanding system input signal is processed periodically with period T_i and results of processing are revealed in time instants $t = n \cdot t_i$ for $n = 1, 2 \ldots$

Let us assume that a "smooth" signal for $n \cdot T_i \le t \le (n+1) \cdot T_i$ is given by formula:

$$x(n \cdot T_i + t) = x(n \cdot T_i) + \frac{t}{T_i}\{x[(n+1) \cdot T_i] - x[n \cdot T_i]\} \quad 0 \le t \le T_i\ . \tag{1}$$

In fact it is :

$$x(n \cdot T_i + t) = x(n \cdot T_i) + \frac{t}{T_i} \cdot \{x[(n+1) \cdot T_i] - x[n \cdot T_i]\} + \sum_{r=1}^{m} A(r, n) \cdot \sin\left(\frac{\pi \cdot r \cdot t}{T_i}\right), \tag{2}$$

where $A(r, n)$ – amplitude of the component of period T_i / r for the interval with consecutive number n. Under the assumption that signal $x(n \cdot T_i + t)$ is smooth (1) the existence of this component entails an error. The condition for correct performance is error minimization via optimal selection of the converter parameters. Those components of (2) which are absent in (1) are responsible for errors.

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The operation realized by the considered evolving system can be described as follows: the dynamic converter, represented by transfer function *K*(*s*) and corresponding impulse response *k*(*t*), processes the signal *x*(*t*). The processing starts at the moment $t = n \cdot T_i$. The initial conditions of the converter at moments $t = n \cdot T_i$ are always equal to zero. The result of processing for $t = (n + 1)T_i$ is revealed (recorded) and the new, consecutive phase of processing starts. So called "controlled integrators" [2] and summers with possibility of tuning of gains for signals representing addends can be used as components of a system realizing the operation *K*(*s*). The above components have to be controlled by clock impulses generated with period T_i (periodic zeroing of initial conditions) .

2. Processing of the input signal

For zero initial conditions one obtains the convolution formula:

$$y[(n+1)\cdot T_i] = \int_0^{T_i} x(n\cdot T_i + t - v)\cdot k(v)\cdot dv \tag{3}$$

for each time range $n \cdot T_i \le t \le (n+1)T_i$. Using (2) one can transform (3) to the form:

$$y[(n+1)\cdot T_i] = x[n\cdot T_i]\cdot h(T_i) + \{x[(n+1)\cdot T_i] - x[n\cdot T_i]\}\cdot H(T_i) + \sum_{r=1}^{m} A(r,n)\cdot S(r,T_i), \tag{4}$$

where:

$$h(T_i) = \int_0^{T_i} k(v)\cdot dv, \quad H(T_i) = \frac{1}{T_i}\cdot\int_0^{T_i} h(v)\cdot dv,$$
$$S(r,T_i) = \int_0^{T_i} k(\tau - v)\cdot \sin\left(\frac{\pi\cdot r\cdot v}{T_i}\right)\cdot dv. \tag{5}$$

The above means that for a smooth signal is $A(r,n) = 0$ and

$$y[(n+1)\cdot T_i] = x[n\cdot T_i]\cdot h(T_i) + \{x[(n+1)\cdot T_i] - x[n\cdot T_i]\}\cdot H(T_i). \tag{6}$$

The (6) holds even for discontinuities of *x*(*t*) for $t = n \cdot T_i$. The coefficients $S(r,T_i)$ define the sensitivities of operation in respect to non-smooth components of *x*(*t*). The error caused by non-smooth components is defined by the last element of formula (4), expressed by sum of products $A(r,n)S(r,T_i)$ [3].

3. Realization of the measurand

If measurand $y_m[(n+1)\cdot T_i]$ represents a linear operation on signal *x*(*t*), like follow-up action, integration, differentiation, delay, weighted average, *etc*., then the result of the operation can be expressed as a linear combination of values:

$$y_m[(n+1)\cdot T_i] = C_1 \cdot x[nT_i] + C_2 \cdot x[(n+1)\cdot T_i]. \tag{7}$$

The (7) holds for smooth signals given by (1). If we admit that result (6) can be scaled according to formula $y[(n+1)\cdot T_i] = p \cdot y_m[(n+1)\cdot T_i]$, where *p* – scaling coefficient, then the following condition can be formulated:

$$\frac{h(T_i)-H(T_i)}{H(T_i)}=\frac{C_1}{C_2} \tag{8}$$

Formula (8) defines the ratio $h(T_i)/H(T_i)$. Assuming the type of the transfer function *K*(*s*) of the converter used to realize the system one should choose such parameters of *K*(*s*) and time T_i that condition (8) is fulfilled. The curves:

$$\begin{aligned} Y_1(t) &= \frac{h(t)}{H(t)} \\ Y_2(t) &= h(t) \end{aligned} \tag{9}$$

make the necessary choice easier. Basing on the curve $Y_1(t)$ one can chose parameters of the transfer function and time T_i fulfilling condition (8) – sometimes there are many solutions. The value of *p* can be determined using curve $Y_2(t)$. If one obtains a set of solutions, then the final choice has to be done. The final solution should be chosen from solutions of (4,5) for various values of parameters of the transfer function *K*(*s*) and times T_i in such a way it (the solution) minimizes errors caused by lack of smoothness of *x*(*t*), *i.e*. values of $\frac{1}{p}\cdot S(r,T_i)$. It means that the best solution ought to guarantee the minimum of $\frac{1}{p}\cdot S(r,T_i)$. Maximal shortening of T_i seems to be most decisive, because time T_i influences the values *A*(*r*, *n*), especially the value *A*(1, *n*), which is usually inversely proportional to T_i^2.

4. Examples

Let us consider the following three types of linear operations:

– Follow-up action:

$$y[(n+1)T_i]=x[(n+1)T_i]. \tag{10}$$

The above operation requires $h(T_i)/H(T_i)=1$.

– Integration:

$$y[(n+1)\cdot T_i]=\frac{1}{T_i}\cdot\int_0^{T_i} x(n\cdot T_i+t)\cdot dt=\frac{1}{2}\{x[n\cdot T_i]+x[(n+1)\cdot T]\}. \tag{11}$$

For that operation one should chose $h(T_i)/H(T_i)=2$.

– Differentiation:

$$y[(n+1)\cdot T_i]=T_i\cdot\frac{d}{dt}x(n\cdot T_i+t)=x[(n+1)\cdot T_i]-x[n\cdot T_i]. \tag{12}$$

Now the relation $h(T_i)/H(T_i)=0$ ought to be attained.

– Delaying for time relation delay:

$$t_0=q\cdot T_i,\quad y[(n+1)\cdot T_i]=x[(n+1-q)T_i]. \tag{13}$$

The correct proportion is $h(T_i)/H(T_i)=1/(1-q)$.

– Weighted averaging for exemplary weight function $\omega(t)=\exp(a\cdot t/T_i)$:

$$y\left[(n+1)\right]=\int_0^{T_i} x(n\cdot T_i+t)\cdot\omega(t)\cdot dt\cdot\left\{\int_0^{T_i}\omega(t)\cdot dt\right\}^{-1}=$$
$$=x\left[nT_i\right]\cdot\left(1+\frac{1}{a}-\frac{e^a}{e^{a-1}}\right)+x\left[(n+1)\cdot T_i\right]\cdot\left(\frac{e^a}{e^a-1}-\frac{1}{a}\right). \tag{14}$$

Now $h(T_i)/H(T_i)=\dfrac{a\cdot(e^a-1)}{a\cdot e^a-e^a+1}$ should be fulfilled.

It is obvious that each of the operations described above can be treated as a particular case of formula (7).

Let us assume that the operations mentioned above have to be realized by means of a converter described by transfer function:

$$K(s)=\frac{k}{1+2\cdot B\cdot\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}, \tag{15}$$

where parameters k, B i ω_0 are properly chosen and the converter is stable (*i.e.* $B\geq 0, \omega_0\geq 0$). The curves $Y_1(t\cdot\omega_0)$ and $-\mathrm{Y}_2(t\cdot\omega_0)$ for B =0 (case 1), B = 0.2 (case 2), B = 0.35 (case 3), B = 0.65 (case 4) and B = 3 (case 5) are shown in Fig. 1. For example, the possibilities of realization of follow-up action (a) for B = 0 as well as integration (b) for B = 0 and B = 3 are transparently indicated in Fig. 1. Table 1 contains the set of results and respective values of coefficients $S(r, T_i)$, for r = 1,2,…5, obtained by simulations using a typical computer program solving differential equations and relations (3) and (5) written in the form of such equations with taking into account the form (15) of transfer functions. The results of Table 1 are not optimal but only illustrate the possibility of obtaining some forms of the measurand. No optimization algorithm for the converter is presented and its creation does not seem to be a simple task. Most significant is the tendency to shorten the time T_i and to lower the values $S(r, T_i)/p$

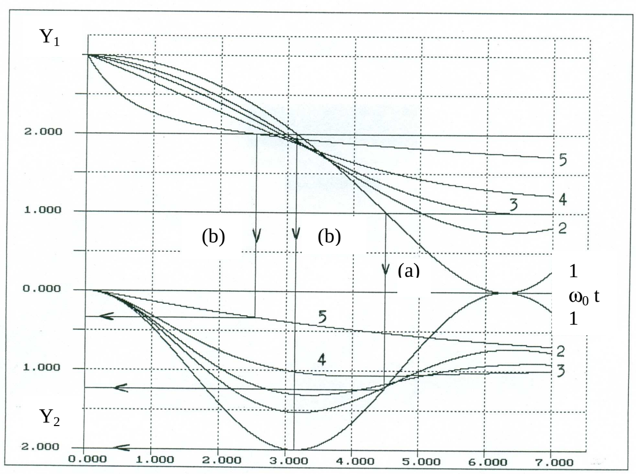

Fig. 1. The curves $Y_1(t\cdot\omega_0)$ and $Y_2(t\cdot\omega_0)$ for transfer function (15) and five values of B = 0,0.2,0.35.0.65,3. The bigger B the bigger the digit indicating the respective curve.

Table 1. The parameters of transfer function (15) chosen for realization of follow-up action, integration, differentiation and respective values of coefficients $S(r, T_i)$.

Lp.	B	$\omega_0 \cdot T_i$	$\frac{h(T_i)}{H(T_i)}$	k	$S(1, T_i)$	$S(2, T_i)$	$S(3, T_i)$	$S(4, T_i)$	$S(5, T_i)$
1	0.00	4.48	1	0.977	1.340	-1.420	-0.600	-0.040	-0.301
2		6.29	0	1.000	0.065	-3.140	-0016	0.004	-0.008
3		3.13	2	0.500	1.570	0.004	0.004	0.000	0.003
4	0.20	5.03	1	1.000	0.894	-1.350	-0.137	-0.205	-0.108
5		3.00	2	0.658	1.167	-0.042	0.047	0.018	0.024
6	0.35	6.70	1	1.106	0.501	-1.330	0.566	-0.152	0.084
7		2.90	2	0.787	0.959	-0.044	0.069	0.025	0.034
8	0.65	2.99	$\frac{1}{0.488}$	1.000	0.740	-0.089	0.086	0.013	0.037
9		7.10	$\frac{1}{0.819}$	1.000	0.591	-0.830	0.484	-0.238	0.119
10		2.78	2	1.037	0.707	-0.045	0.087	0.025	0.040
11	3.00	2.53	2	3.000	0.224	-0.014	0.064	0.002	0.032
12	-0.27	4.64	1	1.103	2.090	-2.700	-2.14	-1.410	-1.120
13		3.47	2	0.298	2.760	-0.166	-0.17	-0.116	-0.116
14		7.76	$\frac{-1 \cdot 19}{0}$	-0.84	-4.090	-6.690	5.06	6.100	4.400

The data gathered in Table 1 incline to the formulation of the following conclusions:

- The possibilities of follow-up operations on smooth signal ($h(T_i) / H(T_i) = 1$) appear for a damping coefficient $B < 0.35$. The statement: "the smaller B the shorter time T_i" applies to the considered case. For negative values of B one can observe higher sensitivities to non-smooth components of the input signal. It makes that an application of converters with $B < 0$ seems to be irrational.
- For "classic" value of damping coefficient $B < 0.65$, which usually characterizes dynamics of recorders, we cannot fulfill the requirement $h(T_i) / H(T_i) = 1$ for times T_i as short as those proper for operation of dot-recorders [4]. In the considered case we can speak rather about an averaging operation or operation of delaying the input signal. The above conclusion corresponds with the thesis formulated by dynamical measurement science [5] and respective parameters q and a can be determined from formula (13) or (14). For line 8 in Table 1 we obtain a delay time $t_0 = 1.53 / \omega_0$. For line 9 in Table 1 we obtain $t_0 = 1.29/\omega_0$. So, for the same B the delay depends on T_i and that effect ought to be explained.
- The big coefficients B are necessary for integration. There are two reasons supporting the previous statement: short times T_i and small sensitivities to non-smoothness of input signal. We can easily observe that for big B the transfer function (15) becomes the transfer function of an integrating converter. This can be treated as an explanation of the obtained result.
- Line 2 in Table 1 corresponds with differentiation of a smooth signal for relatively long time T_i and high sensitivity to non-smooth component referring to $r = 2$.
- Line 14 in Table 1 applies to the operation of delaying of the input signal by delay time T_i. Operation "generated" by line 14 is extremely sensitive to non-smooth components of the input signal. The application of the last two solutions in practice seems to be at least discussable, however they seem to be very interesting, because we are able to realize them by means of a converter with completely different properties.

5. The effect of delaying of a smooth signal

Expanding the integrand of the convolution integral in Taylor's series we obtain the following model of the input-output relation for a smooth input signal $x(t)$ [6]:

$$y(t) = h(t) \cdot x\{t - t_0(t)\},$$
$$t_0(t) = \int_0^t t \cdot k(t) \cdot dt \cdot \left\{ \int_0^t k(t) \cdot dt \right\}^{-1}, \tag{16}$$

where: $\dfrac{t_0(t)}{t} = 1 - \dfrac{\frac{1}{t} \cdot \int_0^t h(t) \cdot dt}{h(t)}$.

Hence, using (13), one obtains:

$$\frac{t_0(T_i)}{T_i} = 1 - \frac{H(T_i)}{h(T_i)} = q\,. \tag{17}$$

Finally, we obtain:

$$t_0(T_i) = q \cdot T_i \tag{18}$$

exactly like it was assumed in formula (13). Using curves $Y_1(t)$, $Y_2(t)$ and taking into account that $h(T_i) / H(T_i) = 1/1–q$ one can determine $\omega_0 T_i, \omega_0 t_0$ and k as a function of B, q, if the converter is represented by transfer function (15). For converter (15) and $0 < B < 1$ one can chose q belonging to the range 0.2, …..0.6. The curves 0.1 $\omega_0 T_i$,(B), $\omega_0 T_0(B)$, $\omega_0 T_0$ (B) and $k^{-1}(B)$ are shown in Fig. 2. Additionally, the relation for $t = \infty$:

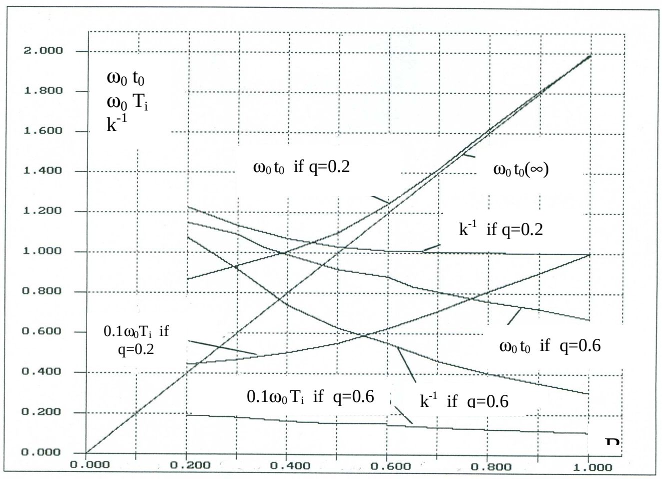

Fig. 2. The curves $\omega_0 t_0(B)$, $\omega_0 t_0(\infty)$, 0.1 $T_i(B)$ $\omega_0 t_0(B)$, and k^{-1} (B) obtained for q equal to 0.2 and 0.6.

$$\lim \omega_0 \cdot t_0(t) = 2 \cdot B = \omega_0 t_0(\infty), \tag{19}$$

which holds for (15) is shown in Fig. 2 as well.

We can observe that $T_i(B)$ and $t_0(B)$ for q = 0.2 increase, if B increases too and for big B curve $t_o(B)$ asymptotically approaches the curve $t_0(\infty)$. For B < 0.8 and small t_0, delays $t_0(t)$ given by (16) are bigger than $t_0(\infty)$. Thus, there is accordance with data in Table 1. For q = 0.6 delays $T_i(B)$ and $t_0(B)$ decrease and the required $k(B)$ increases if B increases. For example, if B = 3, then $T_i(B)$ and $\omega_0 \cdot T = 0.49$, $\omega_0 \cdot t_0 = 0.3$ and k = 18.4. The small delays, if other deformations are absent, can be acceptable from the point of view presented by dynamical measurement science [7]. The product $T_i \cdot \omega_0$ for the currently considered case is substantially smaller comparing it to referring value for "classic" choice of the damping factor, *i.e.* for B = 0.2 (see data in Table 1 for follow-up action). The curves representing signal $x(t) = \cos 0.5 \cdot t$, its perfectly delayed form $x_1(t) = \cos 0.5 \cdot (t - 0.3)$and the output signal $y(t)$ of the expanding system are shown in Fig. 3. We can observe good "convergence" of values $y(n \cdot T_i)$ to $x_1(t)$. It should be mentioned that an application of a classic linear converter with B = 0.65 leads us to a four times greater delay (then the delay time is 1.3 s).

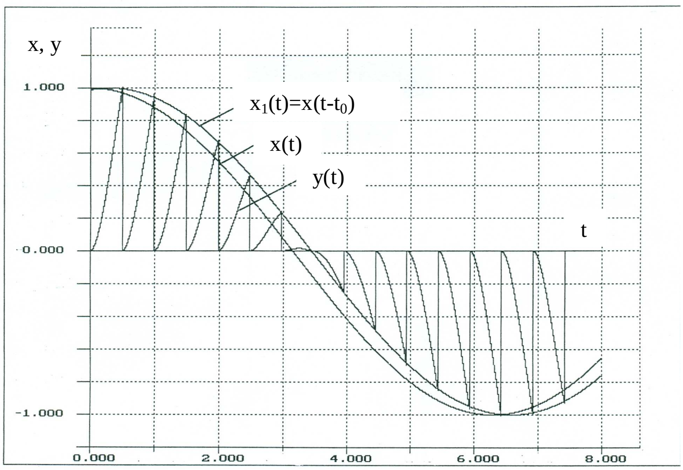

Fig. 3. The input signal $x(t) = \cos 0.5t$, the perfectly delayed input signal $x_1(t_0)$ and the output signal $y(t)$ of an expanding system based on transducer (15) with B = 3.

6. Summary

The considered type of expanding systems can realize many linear operations on smooth signals. The non-typical forms of transfer functions can be used for realization of required operations. The results referring to follow-up action, if small delays are acceptable, are better than those obtained for linear converters with optimal parameters representing their dynamics.

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