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COMPUTATION OF RECONSTRUCTION FUNCTION FOR SAMPLES IN SHIFT-INVARIANT SPACES

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Abstract

We address the problem of reconstructing a class of sampled signals which is a member of shift-invariant spaces. In the traditional method, the reconstruction was obtained by first processing the samples by a digital correction filter, then forming linear combinations of generated functions shifted with period *T*. In order to eliminate the digital correction filter, we propose a computational approach to the reconstruction function. The reconstruction was directly acquired by forming linear combinations of a set of reconstruction functions. The key idea is to obtain a matrix equation by means of oblique frame theory. The reconstruction functions are obtained by solving the matrix equation. Finally, the computational approach is applied, respectively, to reconstruction of a digitizer which samples the signal by derivative sampling or periodically non-uniform sampling technology. The results show that the method is effective.

Keywords: Hilbert space, shift-invariant spaces, sampling, frame, reconstruction function.

1. Introduction

A signal class that plays an important role in sampling theory is signals in shift-invariant (SI) spaces [1, 2]. Such functions can be expressed as linear combinations of shifts of a set of generators with period *T* [3, 4, 5]. This model encompasses many signals used in measure, instrument, communication and signal processing. For example, the set of bandlimited functions is SI with a single generator. Other examples include splines [6, 7] and pulse amplitude modulation in communications. Using multiple generators, a larger set of signals can be described, such as multiband functions [8, 9, 10]. Sampling theories similar to the Shannon theorem can be developed for this signal class, which allows to sample and reconstruct such functions using a broad variety of filters.

Recently, in order to improve the sampling speed, Papoulis' theory is usually applied to oscilloscope and digitizer. We have extended the scope of Papoulis' theory by introducing a formal distinction between the input space and the reconstruction space. The sampling can then be represented as the inner products of the input signal with a set of sampling vectors, which span the sampling space *S*. Examples include multi-resolution [11] spline decompositions [7]. And the reconstruction is obtained by forming linear combinations of a set of reconstruction vectors that span a space *W*. The reconstruction was obtained by first processing the samples by a digital correction filter [12, 13], then forming linear combinations of a set of reconstruction vectors that span a space *W*. In order to eliminate the digital correction filter, we study the sampling reconstruction in such a case that the samples come from a known shift-invariant space, and present a computing method of the reconstruction function from the oblique frame theory. The reconstruction was directly acquired by forming linear combinations of a set of reconstruction functions. Finally, the computational approach

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is proved by reconstruction of a digitizer which samples the signal by derivative sampling or periodically non-uniform sampling.

The paper is organized as follows. Sampling in shift-invariant spaces that we treat in the paper is introduced in Section 2. Section 3 presents the oblique frame theory. In Section 4 we propose a new computational approach to the reconstruction function using the oblique frame theory. Experiment results are demonstrated in Section 5.

2. Sampling for a shift-invariant spaces

In this paper, we consider more general shift-invariant spaces, generated by L functions $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$. A finitely-generated shift-invariant subspace in l^2 ($\sum_{i=0}^{L-1}\sum_{n\in Z} c_i[n]^2 < +\infty$) is defined as:

$$W = \left\{ x(t) = \sum_{i=0}^{L-1}\sum_{n\in Z} c_i[n]\varphi_i(t-nT) : \{c_i[n]\} \in l^2 \right\}.$$

The functions $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$ are referred to as the generators of W. In the Fourier domain, we can represent any $x(t) \in W$ as:

$$X(\omega) = \sum_{i=0}^{L-1} C_i(\omega)\Psi(\omega), \tag{1}$$

where: $C_i(\omega) = c_i[k]e^{-j\omega k}$.

Our only restriction on the choice of the generating function sequence $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$ is that W is a closed subspace of l^2, with $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$ as its Riesz basis. In other words, there must exist two constants A and B $(0 < A \le B < \infty)$, such that:

$$A\sum_{i=0}^{L-1}\sum_{n\in Z}|c_i[n]|^2 \le \left\| \sum_{i=0}^{L-1}\sum_{n\in Z} c_i[n]\varphi_i(t-nT) \right\|^2 \le B\sum_{i=0}^{L-1}\sum_{n\in Z}|c_i[n]|^2 .$$

Since $x(t)$ lies in a space generated by L functions, it makes sense to sample it with m filters $\{s_i(t)\}_{i=0,1,\ldots,L-1}$, as it is on the left-hand side of Fig. 1. The samples are given by:

$$d_k[n] = \langle s_k(t-nT), x(t) \rangle = \int_{-\infty}^{+\infty} s_k(t-nT)^* x(t)dt. \tag{2}$$

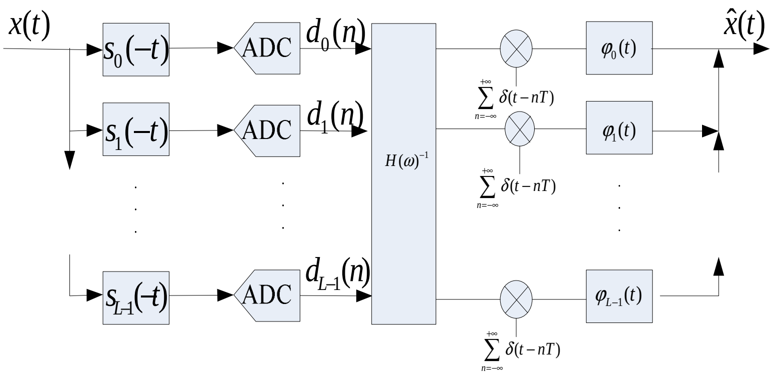

Fig. 1. Sampling and reconstruction in shift-invariant spaces.

At the same time, the choice of the collection $\{s_i(t-nT)\}_{k=0,1,\ldots,L-1,n\in Z}$ forms a frame for *W*. In other words, there must exist two constants *A* and *B* $(0<A\leq B<\infty)$ for every $x(t)\in W$, such that:

$$A\|x(t)\|^2 \leq \sum_{i=0}^{L-1}\sum_{n\in Z}\left|\langle x(t), s_i(t-nT)\rangle\right|^2 \leq B\|x(t)\|^2 .$$

If $0<A=B<\infty$, the frame is said to be an exact frame. An exact frame is a Riesz basis. We define the sampling space:

$$S=\left\{\sum_{i=0}^{L-1}\sum_{n\in Z}c_i[n]s_i(t-nT): c_i[n]\in l^2\right\}.$$

Taking the Fourier transform of $d_i(n)$ in Fig. 1, we have:

$$\begin{aligned} D_i(\omega) &= \sum_{n=-\infty}^{+\infty}\sum_{k=0}^{L-1}C_k(\omega+2\pi n)\psi_k(\omega+2\pi n)S_i(\omega+2\pi n) \\ &= \sum_{k=1}^{L}C_k(\omega)\sum_{n=-\infty}^{+\infty}\psi_k(\omega+2\pi n)S_i(\omega+2\pi n) \qquad i=0,1,\ldots,L-1, \end{aligned} \tag{3}$$

where: $C_k(\omega)$ and $D_i(\omega)$ are the discrete-time Fourier transform of $c_k(n)$ and $d_i(n)$ $\psi_k(\omega)$ and $S_i(\omega)$ are the Fourier transform of $\varphi_k(t)$ and $s_i(-t)$, respectively.

This leads to a compact relation between the sampling data: $d(n)=\begin{pmatrix}d_0(n) & d_1(n) & \ldots & d_{L-1}(n)\end{pmatrix}^T$ and coefficients $c(n)=\begin{pmatrix}c_0(n) & c_1(n) & \ldots & c_{L-1}(n)\end{pmatrix}^T$ of via a matrix-vector multiplication in the Fourier domain:

$$H(\omega)C(\omega)=D(\omega), \tag{4}$$

where:

– $H(\omega)=\begin{pmatrix} h_{0,0}(\omega) & h_{0,1}(\omega) & \ldots & h_{0,L-1}(\omega) \\ h_{1,0}(\omega) & h_{1,1}(\omega) & \ldots & h_{1,L-1}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ h_{L-1,0}(\omega) & h_{L-1,1}(\omega) & \ldots & h_{L-1,L-1}(\omega) \end{pmatrix}$;

– $h_{i,k}(\omega)=\sum_{n=-\infty}^{+\infty}S_i(\omega+2\pi n)\psi_k(\omega+2\pi n_s)$;

– $D(\omega)=\begin{pmatrix}D_0(\omega) & D_1(\omega) & \ldots & D_{L-1}(\omega)\end{pmatrix}^T$;

– $C(\omega)=\begin{pmatrix}C_0(\omega) & C_1(\omega) & \ldots & C_{L-1}(\omega)\end{pmatrix}^T$.

As illustrated in Fig. 1, the reconstruction was obtained by first processing the samples by a digital correction filter $H(\omega)^{-1}$, then forming linear combinations of generated functions $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$ shifted with period *T*. In order to eliminate the digital correction filter we propose that the reconstruction is directly acquired by forming linear combinations of a set of reconstruction functions $\{g_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$. The sampling system is schematically represented in Fig. 2.

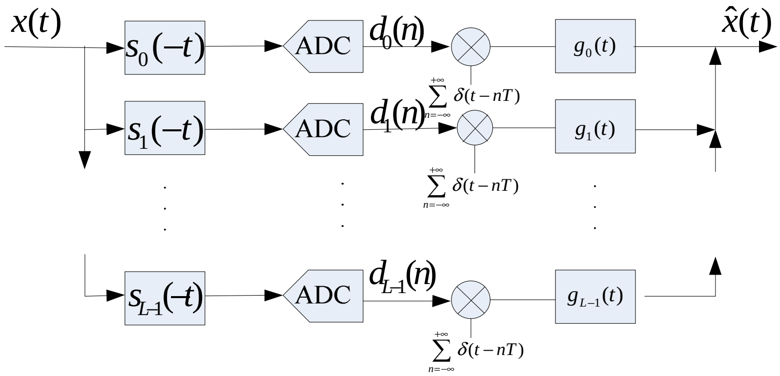

Fig. 2. Improved sampling and reconstruction in shift-invariant spaces.

3. Oblique frame theory

Let us now simply introduce the oblique frame theory. A function sequence $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$ is called an oblique frame of subspace W if there are two constants A and B $(0<A\le B<\infty)$ such that:

$$A\|x(t)\|^2 \le \sum_{i=0}^{L-1}\sum_{n\in Z}\left|\langle x(t), s_i(t-nT)\rangle\right|^2 \le B\|x(t)\|^2$$

holds for any $x(t)\in W$. An exact frame is a Riesz basis. Obviously, a Riesz basis is also a frame. For any oblique frame $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$ of W, there exists a so-called oblique dual frame $\{\tilde{s}_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$ such that:

$$x(t)=\sum_{i=0}^{L-1}\sum_{n\in Z}\langle x(t), s_i(t-nT)\rangle \tilde{s}_i(t-nT) \tag{5}$$

holds for any $x(t)\in W$. Note the definition of an oblique frame operator is:

$$T(x(t))=\sum_{i=0}^{L-1}\sum_{n\in Z}\langle x(t), s_i(t-nT)\rangle s_i(t-nT). \tag{6}$$

It is easy to see that the function sequence $\{T^{-1}(s_i(t-nT))\}_{i=0,1,\ldots,L-1;n\in Z}$ is an oblique dual frame of the oblique frame $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$.

The scalar sequence $\langle x(t), s_i(t-nT)\rangle_{i=0,1,\ldots,L-1,n\in Z}$ is called a moment sequence of the function $x(t)\in W$ to the frame $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$. Let $f(t)=\sum_{i=0}^{L-1}\sum_{m\in Z} c_i[m]s_i(t-m)$.

If the scalar sequence $\{c_i(n)\}_{i=0,1,\ldots,L-1,n\in Z}$ is a moment sequence of a function to the frame $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$, then it must be $c_i(n)=\langle T^{-1}(f(t)), s_i(t-nT)\rangle$. This follows from the fact that $c_i(n)=\langle x(t), s_i(t-nT)\rangle$, for some function $x(t)\in W$ and:

$$T^{-1}(f(t))=\sum_{i=0}^{L-1}\sum_{n\in Z}\langle x(t), s_i(t-nT)\rangle \tilde{s}_i(t-nT)=x(t). \tag{7}$$

4. Derivation of the reconstruction function

The choice of the generating function sequence $\{\varphi_i(t)\}_{i=0,1,\ldots,L-1}$ is that W is a closed subspace of l^2 with as its Riesz basis if and only if [14]:

$$0<\lambda_{\min}\left(\left\|G_\varphi(\omega)\right\|^2\right)\le\lambda_{\max}\left(\left\|G_\varphi(\omega)\right\|^2\right)<\infty,$$

where: $G_\varphi(\omega)=\sum_{k\in Z}\Psi(\omega+2\pi k)\Psi(\omega+2\pi k)^H$, $\Psi(\omega)$ is the Fourier transform of:

$$\varphi(t)=\left(\varphi_0(t)\ \ \varphi_1(t)\ \ \ldots\ \ \varphi_{L-1}(t)\right)^T.$$

Define $\phi(t)$ in W by:

$$\Phi(\omega)=G_\varphi^{-1}(\omega)\Psi(\omega)\,, \tag{8}$$

where $\Phi(\omega)$ is the Fourier transform of $\phi(t)$.

In order to obtain the reconstruction function, define $q_k(nT,t)$ as in (1):

$$q_i(nT,t)=\sum_{m\in Z}(s_i*\phi)^T(nT-mT)\varphi(t-mT)\,. \tag{9}$$

Then, the function sequence $\{q_i(nT,t)\}_{i=0,1,\ldots,L-1,n\in Z}$ is continuous in W. For any function $x(t)\in W$, there is a scalar sequence $\{c_i(n)\}_{i=0,1,\ldots,L-1,n\in Z}$ such that $x(t)=\sum_{i=0}^{L-1}\sum_{n\in Z}c_i[n]\varphi_i(t-nT)$. Following the Parseval identity, we derive:

$$\begin{aligned}
&\left\langle x(t),q_i(nT,t)\right\rangle\\
&=\frac{1}{2\pi}\left\langle X(\omega),Q_i(nT,\omega)\right\rangle\\
&=\frac{1}{2\pi}\left\langle C(\omega)^T\Psi(\omega),\Psi(\omega)^T\sum_{m\in Z}(s_i*\phi)(nT-mT)e^{-j\omega m}\right\rangle\\
&=\frac{1}{2\pi}\left\langle C(\omega)^T\Psi(\omega)\Psi(\omega)^H,\sum_{m\in Z}(s_i*\phi)(nT-mT)e^{-j\omega m}\right\rangle\\
&=\frac{1}{2\pi}\int_0^{2\pi}C(\omega)^T\sum_{k\in Z}\Psi(\omega+2\pi k)\Psi(\omega+2\pi k)^H G_\varphi^{-1}(\omega)\sum_{m\in Z}(s_i*\varphi)(nT-mT)e^{jm\omega}d\omega\\
&=\sum_{m\in Z}(s_i*\sum_{k\in Z}c^T(k)\varphi(t-kT))(nT-mT)\\
&=d_i(n).
\end{aligned} \tag{10}$$

From (10), it implies that $\{q_i(nT,t)\}_{i=0,1,\ldots,L-1,n\in Z}$ is a frame of W. For any $i=0,1,\ldots,L-1,n\in Z$, take the oblique dual function $g_i(t-nT)=T^{-1}(s_i(t-nT))$, where T is the oblique frame operator of the frame $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$. So $\{g_i(nT,t)\}_{i=0,1,\ldots,L-1,n\in Z}$ is an oblique dual frame of $\{s_i(t-nT)\}_{i=0,1,\ldots,L-1,n\in Z}$ in W, such that:

$$\sum_{i=0}^{L-1}\sum_{n\in Z}\left\langle x(t),s_i(t-nT)\right\rangle g_i(t-nT)=x(t).$$

So the frame $\{g_i(nT,t)\}_{i=0,1,\ldots,L-1,n\in Z}$ of the space W is called a reconstruction frame. In a real-world application, we need to know the expression of the reconstruction frame $\{g_i(nT,t)\}_{i=0,1,\ldots,L-1,n\in Z}$. As $q_i(t-nT)=T_q^{-1}(g_i(t-nT))$, we have:

$$q_i(nT,t)=\sum_{k=0}^{L-1}\sum_{m\in Z}\langle q_i(nT,t),q_k(mT,t)\rangle g_k(t-mT). \tag{11}$$

Substituting (9) into the Fourier transform of $q_i(nT,t)$, $Q_i(\omega,t)$ can be rewritten as:

$$\begin{aligned}Q_i(\omega,t)&=\sum_{n\in Z}q_i(nT,t)e^{-j\omega n}\\&=\sum_{n\in Z}\sum_{m\in Z}(s_i*\phi)^T(nT-mT)\varphi(t-mT)e^{-j\omega n}\\&=\sum_{n\in Z}\sum_{m\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\varphi(t-mT)e^{-j\omega m}\\&=\sum_{n\in Z}\sum_{m\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi m)e^{-j(\omega+2\pi l)\tau}.\end{aligned} \tag{12}$$

Let:

$$G_k(\omega)e^{-j\omega t}=\sum_{n\in Z}g_k(t-nT)e^{-j\omega n}. \tag{13}$$

Substituting (13) into the Fourier transform of $\sum_{k=0}^{L-1}\sum_{m\in Z}\langle q_i(nT,t),q_k(mT,t)\rangle g_k(t-mT)$, it can be rewritten as:

$$\begin{aligned}&\sum_{n\in Z}\sum_{k=0}^{L-1}\sum_{m\in Z}\int_{-\infty}^{+\infty}\sum_{l\in Z}(s_i*\phi)^T(nT-lT)\varphi(\tau-lT)\sum_{p\in Z}(s_k*\phi)^T(mT-pT)\varphi(\tau-pT)g_k(t-mT)e^{-j\omega n}d\tau\\&=\sum_{k=0}^{L-1}\sum_{m\in Z}\int_{-\infty}^{+\infty}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\varphi(\tau-lT)\sum_{p\in Z}(s_k*\phi)^T(mT-pT)\varphi(\tau-pT)g_k(t-mT)e^{-j\omega l}d\tau\\&=\sum_{k=0}^{L-1}\sum_{m\in Z}\int_{-\infty}^{+\infty}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi l)\sum_{p\in Z}(s_k*\phi)^T(mT-pT)\varphi(\tau-pT)g_k(t-mT)e^{-j(\omega+2\pi l)\tau}d\tau\\&=\sum_{k=0}^{L-1}\sum_{m\in Z}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi l)\sum_{p\in Z}(s_k*\phi)^T(mT-pT)\Psi(\omega+2\pi l)g_k(t-mT)e^{-j\omega p}\\&=\sum_{k=0}^{L-1}\sum_{m\in Z}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi l)\sum_{p\in Z}\left(S_k(\omega+2\pi p)\Phi^T(\omega+2\pi p)\right)^*\Psi(\omega+2\pi l)g_k(t-mT)e^{-j\omega m}\\&=\sum_{k=0}^{L-1}\sum_{m\in Z}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi l)\sum_{p\in Z}\left(S_k(\omega+2\pi p)\Phi^T(\omega+2\pi p)\right)^*\Psi(\omega+2\pi l)G_k(\omega)e^{-j\omega t}.\end{aligned} \tag{14}$$

Let:

$$A_{i,k}(\omega,t)=\sum_{m\in Z}\sum_{l\in Z}\sum_{n\in Z}S_i(\omega+2\pi n)\Phi^T(\omega+2\pi n)\Psi^*(\omega+2\pi l)\sum_{p\in Z}\left(S_k(\omega+2\pi p)\Phi^T(\omega+2\pi p)\right)^*\Psi(\omega+2\pi l)e^{-j\omega t}.$$

Then we have:

$$Q(\omega,t)=A(\omega,t)G(\omega), \tag{15}$$

where:

– $A(\omega,t)=\begin{pmatrix}A_{0,0}(\omega,t) & A_{0,1}(\omega,t) & \cdots & A_{0,L-1}(\omega,t)\\ A_{1,0}(\omega,t) & A_{1,1}(\omega,t) & \cdots & A_{1,L-1}(\omega,t)\\ \vdots & \vdots & \ddots & \vdots\\ A_{L-1,0}(\omega,t) & A_{L-1,1}(\omega,t) & \cdots & A_{L-1,L-1}(\omega,t)\end{pmatrix}$;

– $G(\omega)=\left(G_0(\omega)\quad G_1(\omega)\quad \cdots\quad G_{L-1}(\omega)\right)^T$;

– $Q(\omega,t)=\left(Q_0(\omega,t)\quad Q_1(\omega,t)\quad \cdots\quad Q_{L-1}(\omega,t)\right)^T$.

From (15), the reconstruction frame $G(\omega)$ is derived by:

$$G(\omega)=A^{-1}(\omega,t)Q(\omega,t). \tag{16}$$

From (13), the generated functions $\{g_k(t)\}$ follow that:

$$\begin{aligned} g_k(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{+\infty} g_k(t-nT)e^{jn\omega}d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G_k(\omega)e^{j\omega t}d\omega \qquad k=0,1,\cdots,L-1. \end{aligned} \tag{17}$$

5. Experiment and analysis

We now apply the computational approach to reconstruction from samples of a digitizer. The digitizer is schematically represented in Fig. 3. In the digitizer, two sampling methods are selected, one is derivative sampling and the other is periodically non-uniform sampling. So we utilize the computational approach of the reconstruction function for sampling in shift-invariant spaces to derivative sampling and periodically non-uniform sampling. In section V-A, we consider the derivative sampling, and in section V-B we provide the periodically non-uniform sampling.

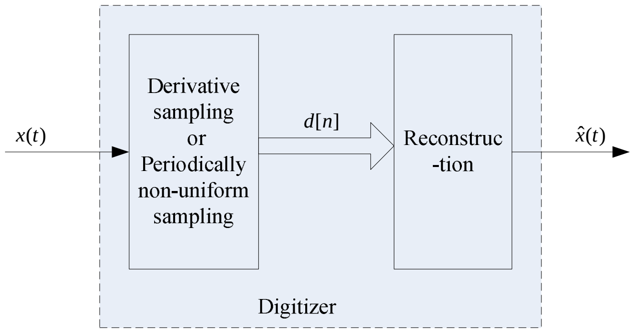

Fig. 3. The structure of a digitizer.

5.1. Derivative sampling

We consider the case L=2. The corresponding analysis filters in the block diagram in the Fig. 1 are $s_0(-t)=\delta(t)$ and $s_1(-t)=\delta'(t)$. The generating functions $\varphi_0(t)$ and $\varphi_1(t)$ of reconstruction space W are given by:

$$\varphi_0(t) = \sin c(\frac{t}{T})\exp(-j2\pi\frac{5}{6T}t), \tag{18}$$

$$\varphi_1(t) = \sin c(\frac{t}{T})\exp(j2\pi\frac{5}{6T}t), \tag{19}$$

where: T is the sampling period.

We easily derive the reconstruction functions by using the oblique frame theory. The reconstruction functions $g_0(t)$ and $g_1(t)$ are expressed by:

$$g_0(t) = \frac{-4\sin(\frac{1}{3}\omega_0 t)+2\sin(\frac{4}{3}\omega_0 t)-2\sin(\frac{2}{3}\omega_0 t)}{3\omega_0 t} - \frac{\cos(\frac{4}{3}\omega_0 t)+\cos(\frac{2}{3}\omega_0 t)-2\cos(\frac{1}{3}\omega_0 t)}{\omega_0^2 t^2},$$

$$g_1(t) = -\frac{\cos(\frac{4}{3}\omega_0 t)+\cos(\frac{2}{3}\omega_0 t)-2\cos(\frac{1}{3}\omega_0 t)}{\omega_0^2 t},$$ where: $\omega_0 = 2\pi/T$.

We sample a continuous-time signal $x(t) = \sin(45\times10^6 t) + \sin(120\times10^6 t)$ using the derivative sampling system with $T = 2\pi/10^8$. Obviously, $x(t) \in W$. The reconstruction signal is shown in Fig. 4. In Fig. 4, the sampling points in the first channel and second channel are marked by "$*$" and "$\bullet$". Fig. 4a shows the first channel reconstruction signal. The second channel reconstruction signal is described in Fig. 4b. The linear combination of two channel reconstruction signals is shown in Fig. 4c. The reconstruction algorithm is proven by comparing Fig. 4c and Fig. 4d, it is shown that the reconstruction algorithm is effective.

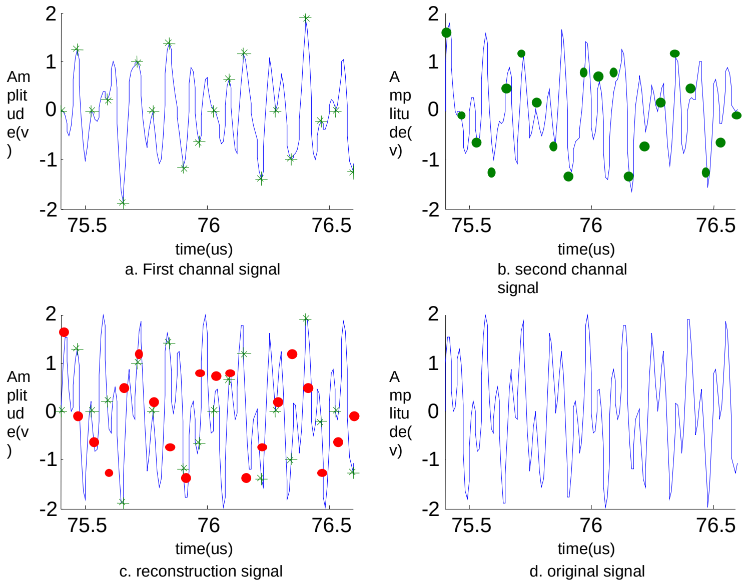

Fig. 4. Reconstruction signal from differential sampling.

5.2. Periodically non-uniform sampling

In this very structured form of periodically non-uniform sampling, the samples are acquired at two distinct locations $\Delta t_0 = 0,\ \Delta t_1 = T/3$ within the basic sampling period *T*. The corresponding analysis filters are $s_0(-t) = \delta(t)$ and $s_1(-t) = \delta(t - \Delta t)$. The generating functions $\varphi_0(t)$ and $\varphi_1(t)$ of reconstruction space *W* are given by (18) and (19).

From (16), the reconstruction functions $g_0(t)$ and $g_1(t)$ are expressed by:

$$g_0(t) = \frac{\sin(\omega_0 t)}{\omega_0 t}\exp(j\frac{5}{6}\omega_0 t) + (e^{-j\omega_0\Delta t} - 1)\frac{\sin(\frac{\omega_0}{6}(t-\Delta t))}{\frac{\omega_0}{6}(t-\Delta t)}\exp(-j\frac{1}{2}\omega_0(t-\Delta t))$$

$$+ (e^{-j2\omega_0\Delta t} - 1)\frac{\sin(\frac{\omega_0}{3}(t-\Delta t))}{\frac{\omega_0}{3}(t-\Delta t)}\exp(-j\omega_0(t-\Delta t)),$$

$$g_1(t) = \frac{1}{(e^{j\omega_0\Delta t}-1)} \frac{\sin(\frac{\omega_0}{6}t)}{\frac{\omega_0}{6}t} \exp(j\frac{1}{2}\omega_0 t) + \frac{1}{(e^{j2\omega_0\Delta t}-1)} \frac{\sin(\frac{\omega_0}{3}t)}{\frac{\omega_0}{3}t} \exp(j\omega_0 t)$$

$$+ \frac{1}{(1-e^{-j\omega_0\Delta t})} \frac{\sin(\frac{\omega_0}{6}(t+\Delta t))}{\frac{\omega_0}{6}(t+\Delta t)} \exp(-j\frac{1}{2}\omega_0(t+\Delta t)) + \frac{1}{(1-e^{j2\omega_0\Delta t})} \frac{\sin(\frac{\omega_0}{3}(t+\Delta t))}{\frac{\omega_0}{3}(t+\Delta t)} \exp(-j\omega_0(t+\Delta t)),$$

where $\omega_0 = 2\pi / T$.

Assuming an input signal $x(t) = 2\sin(5\pi \times 10^8 t/6)\cos(\pi \times 10^8 t/3)$ and the sampling period $T = 1/10^8$, obviously $x(t) \in W$. Fig. 5 shows the reconstruction signal; the sampling points in the first channel and second channel are marked by "$*$" and "$\bullet$", the dotted lines are the imaginary of reconstruction signals. By comparing Fig. 5c and Fig. 5d, the validity of the reconstruction algorithm will be proven too.

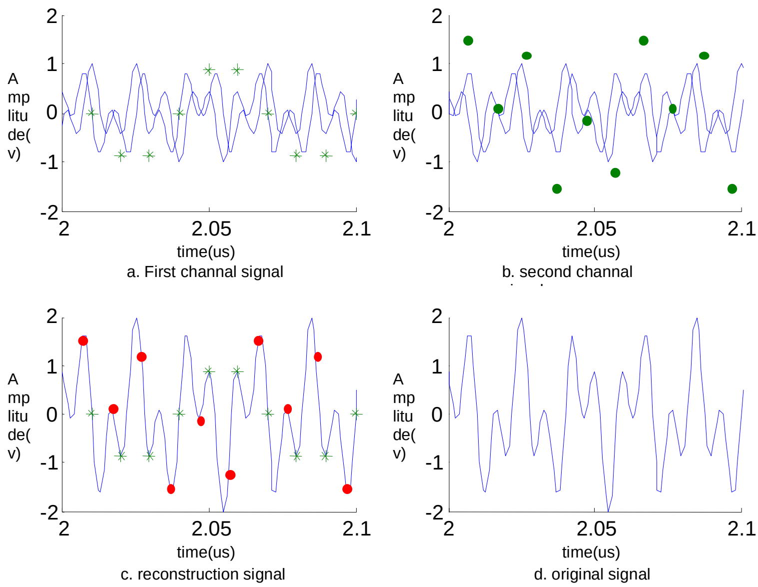

Fig. 5. Reconstruction signal from periodically non-uniform sampling.

6. Conclusion

In this paper, we studied the problem of recovering a signal $x(t)$ in shift-invariant spaces, from L given sets of samples which are modeled as inner products of $x(t)$ with sampling functions $s_i(-t)$, $0 \le i \le L-1$. In the traditional method, the reconstruction was obtained by first processing the samples by a digital correction filter, then forming linear combinations of generated functions shifted with period T. In order to eliminate the digital correction filter, we derive a computational approach to the reconstruction function, which turns computing of reconstruction functions into solving a matrix equation by means of the oblique frame theory. It is ensured that the reconstruction functions can be effectively obtained. Thus the reconstruction was directly acquired by forming linear combinations of a set of reconstruction functions. Finally, the method is verified through reconstruction of periodically non-uniform sampled or derivative sampled signals of a digitizer.

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