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# THE TRANSPORTATION DISTANCE FOR FUZZY DESCRIPTIONS OF MEASUREMENTS

Fuzzy nominal scales were introduced in order to propose a formalism to the representation of empirical quantities by fuzzy subsets of words. This scale proposes a similarity relation and an associated bounded distance that can be used to perform signal processing on fuzzy subsets of words. Due to the limits of this last distance, we studied distances associated to this formalism and proposed a new distance operator named transportation distance. This paper presents the results of these studies.

Keywords: Measurement theory, fuzzy description, fuzzy nominal scale, fuzzy distance

## 1. INTRODUCTION

The introduction of the fuzzy subset theory in the measurement field takes its origin in 1971 in Zadeh's paper [1] that exposes a mechanism of description of a quantity by a fuzzy subset of symbols. Since this paper, the definition of the description process was mainly based on good practices. Most of description processes had useful properties but it was sometimes difficult to justify them. Recently, the link between the quantities and their fuzzy representation was defined in the scale formalism [2]. This new scale is named the fuzzy nominal scale [3, 4].

This approach gives the set of relations and operators that can be used to define equations on symbols such that these equations have a meaning on the set of quantity values. When a fuzzy description is a fuzzy nominal scale, a fuzzy equivalence relation on quantity values is linked to a fuzzy equivalence relation on their representation. This last relation also named similarity relation is used to define a distance between the fuzzy subset of symbols that represent the quantity values. This distance had been used to perform signal processing [5], but is useless to compare symbols that are not related by the similarity relation.

The purpose of this paper is to propose a distance on fuzzy representations linked to a metric on the set of quantity values. With this new distance, a fuzzy nominal scale is enhanced and the set of authorized operators has now this distance as member.

## 2. THE FUZZY SYMBOLISMS

The link between a physical state and its linguistic representation is characterized by a symbolism defined by the triplet $<E, S, R>$ where $E$ is the set of physical states, $S$ is the lexical set used to represent measurement results and $R$ is a relation on $E\times S$. Two mappings can be extracted from this relation: The *description mapping* denoted $D$ associates a subset of $S$ to any item of $E$, and the *meaning mapping* denoted $M$ associates a subset of $E$ to any item of $S$. These two mappings are linked with the following equation.

$$\forall e \in E, \forall s \in S, e \in M(s) \Leftrightarrow s \in D(e). \tag{1}$$

The $R$ relation can be a fuzzy relation. Then, the translation of a physical state into its linguistic representation is called a *fuzzy linguistic description mapping* or simply a *fuzzy description mapping*. It transforms an object $e$ of the set of physical states $E$ into a fuzzy subset of linguistic terms called the *fuzzy description* of $x$. The dual mapping, called the *fuzzy meaning* mapping, associates a fuzzy subset of $E$ to each term $s$ of the lexical Set $S$. This fuzzy subset is the *fuzzy meaning* of $s$. In the paper the fuzzy subsets of linguistic terms also named lexical fuzzy subsets are denoted LFS. This two mappings are also linked:

$$\forall (e, s) \in E \times S, \mu_{M(s)}(e) = \mu_{D(e)}(s) \quad . \tag{2}$$

In [7] it is defined that <$E$, $S$, $R$> is a $\phi$-symbolism if the set of the meanings of the elements of $S$ is a $\phi$-partition of $E$ as defined in [8] and if each meaning is normalized. This paper restricts its investigation to id-symbolisms based on id-partition i.e. on Ruspini partition. The set of all possible LFS obtained by a fuzzy description based on id-symbolism is denoted $F_{id}(S)$. Any LFS respects then the condition:

$$\forall A \in F_{id}(S), \sum_{s \in S} \mu_A(s) = 1 \quad . \tag{3}$$

A fuzzy equivalence relation on the physical states can be associated to any id-symbolism.

$$\forall (x, y) \in E^2, \mu_{\sim}(x, y) = \sum_{s \in S} min(\mu_{M(s)}(x), \mu_{M(s)}(y)) \quad . \tag{4}$$

From this fuzzy equivalence relation and from the relation $R$, the following relation can be simply defined.

$$\forall (A, B) \in F_{id}(S)^2, \mu_{\sim}(A, B) = \sum_{s \in S} min(\mu_A(s), \mu_B(s)) \quad . \tag{5}$$

The symbolism <$E$, $S$, $R$> is then considered as a fuzzy nominal scale.

## 3. CHOICE OF A DISTANCE OPERATOR

The relation used in the id-symbolism can define the distance between LFSs [5].

$$d_{\sim}(A, B) = 1 - \mu_{\sim}(A, B) \quad . \tag{6}$$

This distance is discriminant for LFSs that are at least partially equivalent but is equal to 1 when 2 LFS have an empty intersection. This result is consistent with the absence of distance on the lexical set. This means that the definition of a metric on the set $F_{id}(S)$ needs the definition of a metric on the set $S$. Let $d_S$ be a distance defined on $S$.

### 3.1. Required properties

The fuzzy subset theory proposes a large set of distance operators and the best way to select a distance operator is to list the properties that must be verified.

- The first property is the singleton coincidence: If two LFSs are singleton $\{s_1\}$ and $\{s_2\}$ then the distance between them is equal to the distance between symbols $s_1$ and $s_2$. This property supposes that the distance $d_S$ on $S$ exists.
- The continuity property is verified when the distance is a continuous mapping from $F_{id}(S)\times F_{id}(S)$ to the set of positive numbers.
- The precision property simply imposes that the distance between two LFSs must be a positive real number, and not a fuzzy subset of positive real numbers.
- The consistency property is usually verified by distances on crisp subsets: If $A$, $B$, $C$, $E$ are four subsets of a metric space, $d$ is the distance on this space, and $d_g$ is a distance that generalizes $d$ on subsets, it verifies:

$$\sup_{u \in A, v \in B} d(u, v) \le \inf_{x \in C, y \in E} d(x, y) \\ \Rightarrow d_g(A, B) \le d_g(C, E) \quad (7)$$

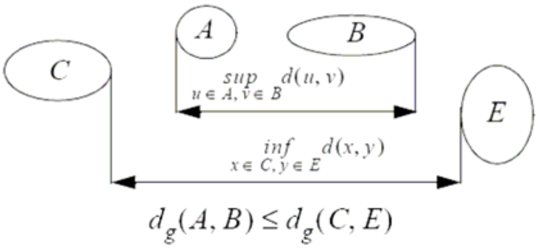


Fig. 1. Consistency property.

The extension of a distance defined in a finite space to a distance defined on the set of the fuzzy subsets of this space was widely studied but, as shown below, no existing distance can be applied to $F_{id}(S)$.

Distances on fuzzy subsets can be classified into the following categories.
- The distances that generalise an existing distance.
- The distances defined from a similarity measure.
- The distances defined with subset operators.
- The distances computed from a symbolic approach.

In our approach, a distance $d_S$ is supposed to be defined on $S$. Then only the first category is investigated.

The generalisation of a distance $d_S$ defined on a finite set $S$, to a distance $d_{F(S)}$ defined on the set of fuzzy subsets of $S$ is a recurrent subject of study. In [9] Bloch proposes four types of generalisation.

### 3.2. The geometrical approach

In this approach, fuzzy subsets in a n-dimensional space are considered as crisp subsets in a (n+1)-dimensional space. This means that the distance between membership degrees has the same semantic than distance in the n- dimensional space. Such hypothesis cannot be justified in our problem and this approach is not kept.

### 3.3. The fuzzification approach

In another approach a distance $D_S$ between crisp subsets is defined from the distance $d_S$. Then the distance $D_S$ is fuzzyfied. In three fuzzifications of the Hausdorff distance are proposed.

$$H^1_{F_{Id}(S)}(F, G) = \int_0^1 H_S(F_\alpha, G_\alpha) d\alpha \quad , \qquad (8)$$

$$H^\infty_{F_{Id}(S)}(F, G) = sup_{\alpha \geq 0} H_S(F_\alpha, G_\alpha) \quad , \qquad (9)$$

$$H^*_{F_{Id}(S)}(F, G) = H_S(F_{1.0}, G_{1.0}) \quad , \qquad (10)$$

where $F_\alpha$ and $G_\alpha$ are the alpha-cuts of $F$ and $G$, and $H_S$ is the Hausdorff distance:

$$H_S(A, B) = max(\max_{a \in A} \min_{b \in B} d_S(a, b), \max_{b \in B} \min_{a \in A} d_S(a, b)) \quad . \qquad (11)$$

$H^\infty_{F_{Id}(S)}$ and $H^*_{F_{Id}(S)}$ do not verify the continuity property, but $H^1_{F_{Id}(S)}$ does. It also verifies the singleton coincidence, but not the consistency property.

### 3.4. The weighting approach

The distance $d_S$ can be generalized with a weighting of membership degrees.

$$d_T(F, G) = \sum_{s_1 \in S} \sum_{s_2 \in S} T(\mu_F(s_1), \mu_G(s_2)) d_S(s_1, s_2) \quad , \qquad (12)$$

where $T$ is a continuous triangular norm.

Such operator does not respect the separation axiom that imposes:

$$d_T(F, G) = 0 \Leftrightarrow F = G \quad , \qquad (13)$$

then it cannot be considered as a distance.

### 3.5. Morphological approach

This last approach is based on morphological operators. For example the Hausdorff distance can be expressed with such operators. The principle is to generalize these operators to fuzzy morphological operators. But most of these generalizations produce fuzzy distances that do not respect the precision property.

### 3.6. A new approach

So a new distance that respects the four properties has been created. This distance is named the transportation distance $d_{tp}$. Its calculation is equivalent to the solution of a mass transportation problem [11]. It is similar to the Wasserstein distance used in probability theory, and can also be considered as a fuzzy version of the Levenshtein distance used to compare strings [13].

## 4. THE TRANSPORTATION DISTANCE

The transportation distance between two LFSs is based on the cost calculation of a set of transformations needed to transform the first LSF to the other. First a family of transformation mappings is defined:

Let $T_{s_i,s_j,x}$ be a mapping on a set $F_{id}(S)$ such that:

$$G = T_{s_i,s_j,x}(F) \Leftrightarrow \begin{array}{l} \mu_G(s_i) = \mu_F(s_i) - x \\ \mu_G(s_j) = \mu_F(s_j) + x \end{array} \quad (14)$$

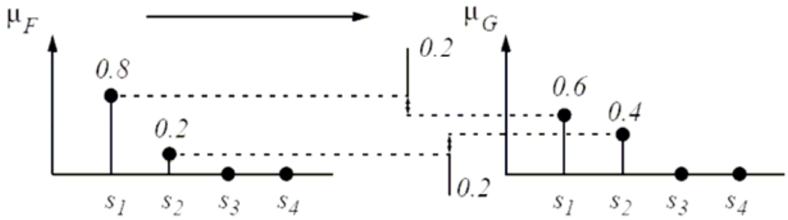


Fig. 2. The mapping $T_{s_1,s_2,0.2}$ with $S = \{s_1, s_2, s_3, s_4\}$.

We demonstrate that any element of $F_{id}(S)$ can be transformed into any other element of $F_{id}(S)$ with the use of a sequence of such transformation mappings.

Let the following sets:

$$S_{F>G} = \{s \in S, \mu_F(s) > \mu_G(s)\},$$

$$S_{F=G} = \{s \in S, \mu_F(s) = \mu_G(s)\}.$$

Proposition 1: Let $S$ be a finite set. Let $F$ and $G$ be 2 elements of the set $F_{id}(S)$. Let $\Delta_s = \mu_F(s) - \mu_G(s)$. The following equality is verified:

$$\sum_{s \in S_{F>G}} \Delta_s = -\sum_{s \in S_{G>F}} \Delta_s . \quad (15)$$

Then the definition of the sequence of transformation mappings is equivalent to the well known linear programming problem named the transportation problem [11]. The problem is to bring a product from a set of $n_1$ sources to a set of $n_2$ destinations. Each source $i$ gives a quantity $x_i$ of product, and each destination receives a quantity $x_j'$ of product. The total given quantity must be equal to the total received quantity:

$$\sum_{i=1}^{n_1} x_i = -\sum_{j=1}^{n_2} x_j' . \quad (16)$$

A solution is to associate to each displacement from a source $i$ to a destination $j$ a quantity $x_{ij}$ of transported product and a displacement unity cost $c_{ij}$. The aim is to find a solution that minimises the total cost:

$$\sum_{i,j} x_{\mathrm{ij}} c_{ij} \quad . \tag{17}$$

Considering the membership degrees as the transported product, the set $S_{F>G}$ as the set of sources and the set $S_{F<G}$ as the set of destinations, a distance can be computed as the total cost of the optimal solution for the transportation of membership degrees.

The transportation distance $d_{tp}$ is defined on $F_{id}(S)$ from distance $d_S$ on the lexical set $S$. The distance $d_{tp}$ is the sum of the costs of each transformation mapping. And the cost of a transformation mapping $T_{s_i,s_j,x_{ij}}$ is equal to: $x_{ij}d_S(s_i, s_j)$.

It is now shown that the transportation distance is a distance, and it verifies the 4 constraints presented before.

For any $F$, $G$, $H \in F_{Id}(S)$, $d_{tp}$ must verify:

$$d_{\mathrm{tp}}(F, G) = 0 \Leftrightarrow F = G, \tag{18}$$

$$d_{\mathrm{tp}}(F, G) = d_{\mathrm{tp}}(G, F), \tag{19}$$

$$d_{\mathrm{tp}}(F, G) + d_{\mathrm{tp}}(G, H) \geq d_{\mathrm{tp}}(F, H). \tag{20}$$

- The relation $F = G$ is equivalent to $S_{F>G} = S_{F<G} = \varnothing$ that is equivalent to $d_{\mathrm{tp}}(F, G) = 0$.
- The symmetry of $d_{\mathrm{tp}}$ is deduced from the symmetry of the transportation problem.
- Finally, $d_{\mathrm{tp}}(F, H)$ is by definition the distance corresponding to the optimal sequence of transformations $T_{s_i,s_j,x}$ that changes $F$ into $H$. Then adding a constraint in order to include $G$ in the set of transformation steps will increase the distance.
- Calculating the distance between singletons $\{l_i\}$ and $\{l_j\}$ using the transportation problem is equivalent to finding a cheaper solution to bring a unity quantity of product from source $i$ to destination $j$. The solution is made of only one transformation mapping $T_{s_i,s_j,1}$. The cost of this solution is $c_{\mathrm{ij}}$ that is equal to the distance $d_S(s_i, s_j)$ then the singleton coincidence is verified.
- The precision and the continuity properties are deduced from the definition of the distance
- The consistency property of $d_{tp}$ is demonstrated below:

  Let $F$, $G$, $H$, $I$ four elements of $F_{Id}(S)$ and $(s_1, s_2, s_3, s_4) \in S^4$ such that:

$$\overline{d_{FG}} \leq \underline{d_{HI}} \quad , \tag{21}$$

where:

$$\underline{d_{HI}} = \inf_{(s_3, s_4) \in \mathrm{supp}(H) \times \mathrm{supp}(I)} d_S(s_3, s_4),$$
$$\overline{d_{FG}} = \sup_{(s_1, s_2) \in \mathrm{supp}(F) \times \mathrm{supp}(G)} d_S(s_1, s_2). \tag{22}$$

and $supp(A)$ is the support of A i.e. the set of lexical terms $s$ such that $\mu_A(s) \neq 0$.

It must be shown that

$$d_{\mathrm{tp}}(F, G) \leq d_{\mathrm{tp}}(H, I) \quad . \tag{23}$$

If $H \cap I \neq \varnothing$, then $\underline{d_{HI}} = 0$ and $\overline{d_{FG}} = 0$. $F$ and $G$ are the same singleton and $d_{tp}(F, G) = 0$. Equation (23) is trivially verified.

If $H \cap I = \varnothing$, the brought quantity associated to the calculation of $d_{tp}(H, I)$ is equal to 1 and $d_{tp}(H, I) \geq \underline{d_{HI}}$ because the transportation distance is then a weighted average of distances that are greater or equal to $\underline{d_{HI}}$. With the same reasoning, $d_{tp}(F, G) \leq \overline{d_{FG}}$. Then, Eq. (21) induces $d_{tp}(F, G) \leq d_{tp}(H, I)$.

## 5. APPLICATION EXAMPLE

In this section, the transportation distance is applied on the hand posture recognition. More details on the application can be found in [12].

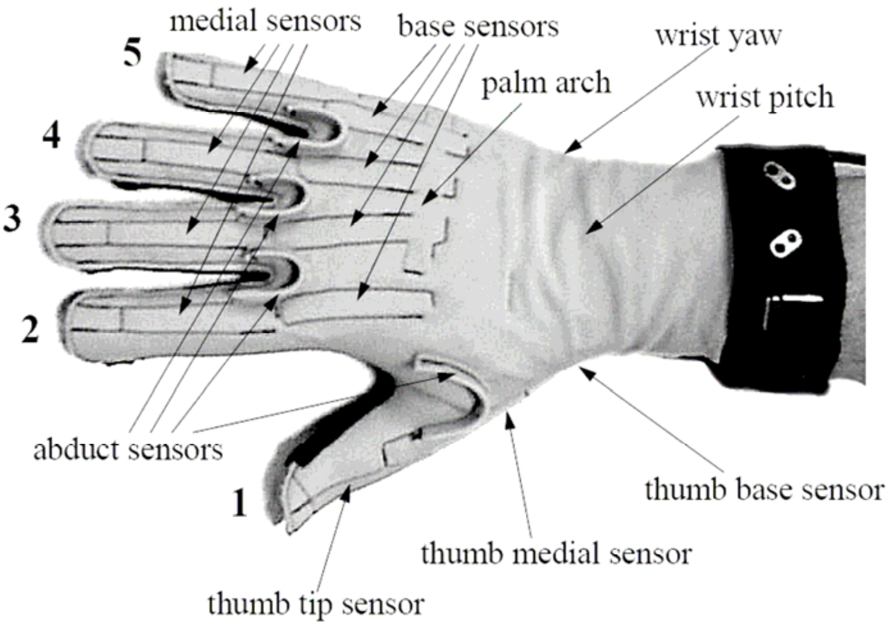


Fig. 3. The 18 sensors of the Cyberglove®.

The hand posture is acquired with the 18 angle sensors of a CyberGlove® (Fig. 3). The finger flexion (except for the thumb) is acquired with two angle sensors: MCP (metacarpal-phalanx angle) and IP (inter phalanx angle). The linguistic description of a finger uses the set $S_{flexion}$ = {*Folded*, *Claw*, *Round*, *Square*, *Straight*} see Fig. 4.

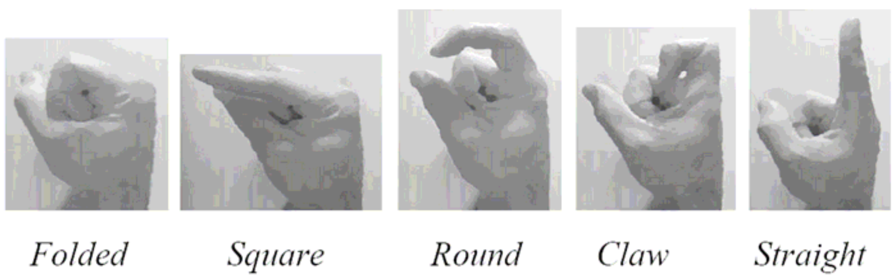


Fig. 4. Words used to describe the finger flexion.

The dataglove gives a numeric representation of the finger flexion as a couple (mcp, ip) $\in \Re^2$. The definition of the fuzzy linguistic description is performed through the definition of the fuzzy meaning of each lexical term. These meanings are fuzzy subsets in $\Re^2$ as shown in example in Fig. 5.

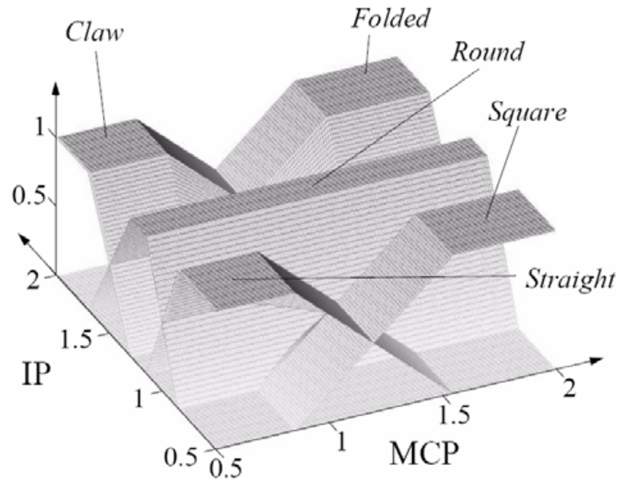


Fig. 5. Meanings of the items of $S_{flexion}$.

The illustration of this new distance is presented with the example of a finger flexion. Considering 2 numeric values of finger flexion $f_1 = (0.5, 0.5)$ and $f_2 = (1.2, 1.0)$ their fuzzy descriptions are shown in Fig. 6.

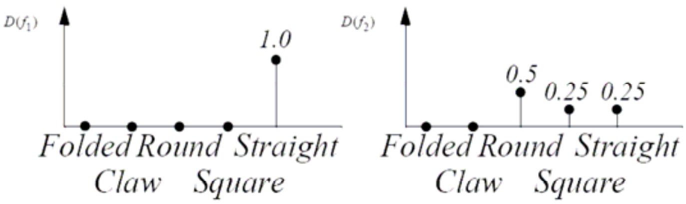


Fig. 6. Descriptions of $f_1$ and $f_2$.

The distance $d_{S_{flexion}}$ is arbitrarily chosen as shown in Table 1. It represents the human knowledge about the description of a finger flexion.

Table 1. Distance $d_{S_{flexion}}$ defined on $S_{flexion}$.

| $d_{S_{flexion}}$ | *Folded* | *Claw* | *Round* | *Square* | *Straight* |
|---|---|---|---|---|---|
| *Folded* | 0 | 1 | 2 | 3 | 4 |
| *Claw* | 1 | 0 | 1 | 2 | 3 |
| *Round* | 2 | 1 | 0 | 1 | 2 |
| *Square* | 3 | 2 | 1 | 0 | 1 |
| *Straight* | 4 | 3 | 2 | 1 | 0 |

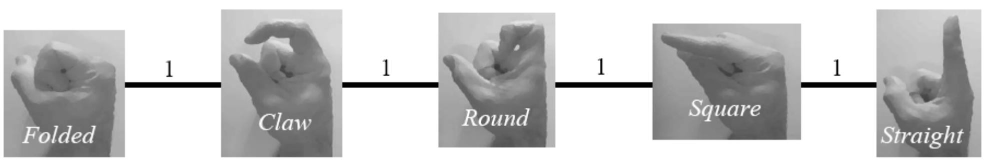


Fig. 7. Graph that represents $d_{S_{flexion}}$.

The distances to the two finger flexions $f_1$ and $f_2$ are calculated for any other finger posture (Figs. 8 and 9). In both figures, five plates correspond to the distance between each term and $f_1$ or $f_2$. The value on each plate is directly connected to the distance $d_{S_{flexion}}$ .

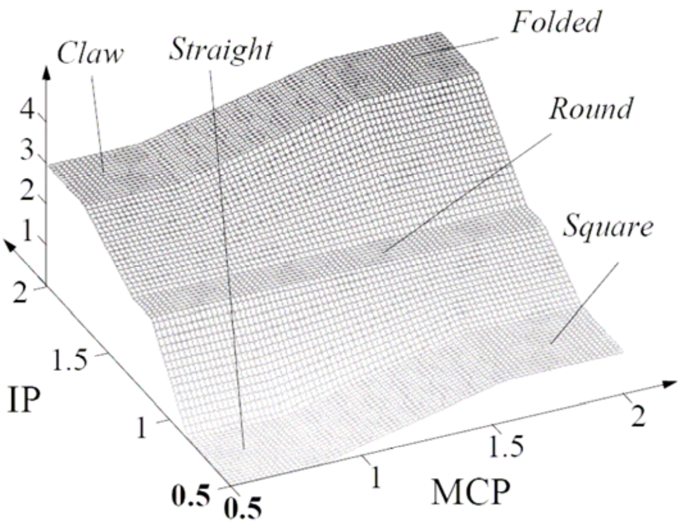


Fig. 8. $d_{tp}(D(f_1), D(g))$ with $f_1 = (0.5, 0.5)$.

In Fig. 9 the values of the plates corresponding to the terms *Square* and *Round* are identical. This means that:

$$d(D(f_2), \{Square\}) = d(D(f_2), \{Round\}) , \tag{24}$$

even if $\mu_{D(f_2)}(Round) \geq \mu_{D(f_2)}(Square)$ as shown in Fig. 6. This result is consistent because the transportation distance takes all the terms into account. In this case $f_2$ is a little bit *Straight:* $\mu_{D(f_2)}(Straight) = 0.25$ .

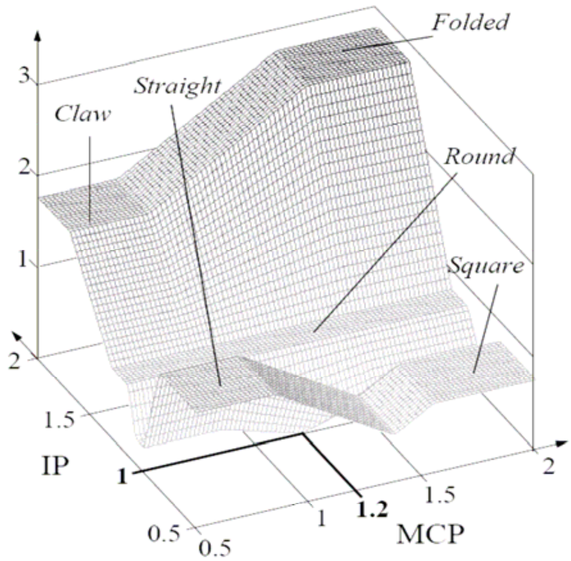


Fig. 9. $d_{tp}(D(f_2), D(g))$ with $f_2 = (1.2, 1.0)$.

## 6. DISCUSSION

Through a fuzzy nominal scale, a link is established between a set of measurement results and a set of fuzzy subsets of linguistic terms. The distance $d_{tp}$ proposed in this paper generalizes the distance $d_S$ defined on a small set of lexical terms such that each term represents a fuzzy subset of measurement results. Then, the distance $d_{tp}$ depends on the definition of the distance $d_S$. In the application example, $d_S$ is arbitrarily chosen. Another choice can be represented by the following graph.

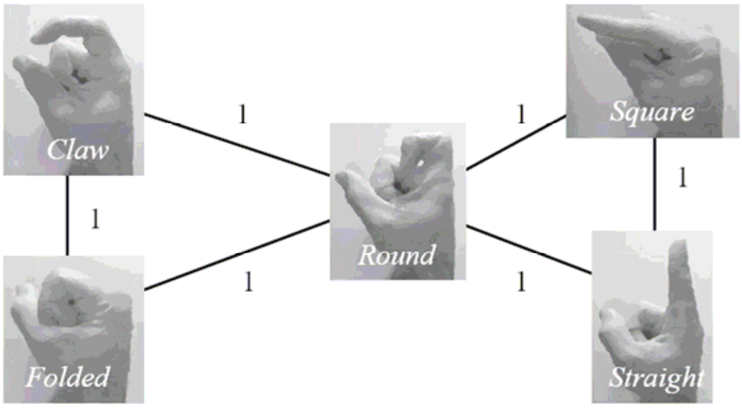


Fig. 10. Another definition for $d_S$.

Such arbitrary choice can be considered as a critical point of this approach. A more objective approach can be based on the scale definition as presented in [3] but the result looks like an arbitrary choice. For example, the distance given in Fig. 10 can be the result of such an objective approach. Actually a fuzzy nominal scale is defined by its ϕ-symbolism that is itself an arbitrary choice even if it respects strict constraints. The definition of such scale, and such distance is not simply driven by the measurement process, but also by the goal of the fusion system that includes the measurement process. In this paper the distance was defined for decision systems based on a set of typical known gestures. In other cases the choice of the distance can be based on statistical data or on the knowledge of physical mechanisms. From a more general point of view, the discussion can be concluded with the proposal that the distance concept is not a part of the empirical world, but a part of its representation.

## 7. CONCLUSION

The scientific process that formalizes the lexical fuzzy subset based descriptions of quantities started with the introduction of fuzzy nominal scales as a bridge between the description process proposed by L. Zadeh, and the scale formalism as presented by L. Finkelstein. The goal of such a process is to create a signal processing formalism where the elementary entities are lexical fuzzy subset based representations of quantities.

With the transportation distance, this paper gives a new tool for processing lexical fuzzy subsets issued from a measurement process. With its four properties: singleton coincidence, continuity, precision and constancy, the transportation distance is a good candidate to perform signal processing on this particular kind of representation issued from a fuzzy description of measurement. In this paper the distance between lexical terms was issued from human knowledge, but it will be possible to extract it from a metric on physical states and from the fuzzy nominal scale. Then the scale will be enhanced in order to bring a metric from the set of physical states to the set of lexical fuzzy subsets. Into the context of a LFS based signal processing, it can now be considered that this one includes a fuzzy equivalence relation, also called similarity, and a distance. A scale based on the association of two similarity was called fuzzy nominal scale, or it can be called a topological scale. We propose that an extension of this scale including the association of two distances be called a metric scale.

The study presented in this paper is a first step in the study of operators that define scales based on ϕ-symbolism. This study needs to be extended to other ϕ-symbolisms and, if applicable, to all of them. But before, the criteria that drive the choice of a ϕ-symbolism need to be objectively defined. The next steps of this research are on one side, defining new signal processing processes based on this metric scale, i.e. that use only the similarity relation and the distance to perform signal processing. On the other side, finding new operators to improve the LFS based signal processing formalism.

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