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RANKINGS AS ORDINAL SCALE MEASUREMENT RESULTS

Rankings (or preference relations, or weak orders) are sometimes considered to be non-empirical, non-objective, low-informative and, in principle, are not worthy to be titled measurements. A purpose of the paper is to demonstrate that the measurement result on the ordinal scale should be an entire (consensus) ranking of *n* objects ranked by *m* properties (or experts, or voters) in order of preference and the ranking is one of points of the weak orders space. The consensus relation that would give an integrative characterization of the initial rankings is one of strict (linear) order relations, which, in some sense, is nearest to every of the initial rankings. A recursive branch and bound measurement procedure for finding the consensus relation is described. An approach to consensus relation uncertainty assessment is discussed.

Keywords: Ordinal scale, weak order, consensus relation, recursive algorithm

1. INTRODUCTION

Rankings are sometimes considered to be non-empirical, non-objective, low-informative and, in principle, are not worthy to be titled measurements [1]. In our opinion, a ranking is a result of measurement on the ordinal scale and is useful to the same extent as any ordinal measurement.

There are a lot of ordinal kind scales in the scope of applied metrology. These are, for example, scales for mineral hardness, earthquake magnitudes, wind force, smell of water, many of scales for different kinds of food quality and many, many others [2]. The point is that measurement results obtained in these scales are frequently treated as some number (score, rank). For example, when measuring hardness on Mohs scale a mineral sample is assigned a number b if it cannot be scratched by standard mineral b, $b = 1, \ldots, 10$, and cannot scratch it. This number is, clearly, only a label and its use in any additive or multiplicative operation is meaningless.

A purpose of the paper is to demonstrate that *the measurement result on the ordinal scale should be the* **entire ranking** *of n objects* and the ranking is one of points of the weak orders space. In this case there appears a possibility to study a structure of the space, to investigate the correlation between rankings and the space cardinality and do many other researches yielding useful information about objects under measurement.

2. ORDINAL MEASUREMENT TREATMENT

As in [3], in this paper ordinal measurement is treated as finding a *consensus ranking* for given initial rankings. In fact, by m properties, n objects can be ranked in order of preference. A single preference relation that would give an integrative characterization of the object properties is one of strict (linear) order relations, which, in some sense, is nearest to every of the initial rankings. Finding the consensus ranking is possible due to introducing a measure of distance between pairs of rankings.

2.1. Rankings

Suppose we have *m rankings* on the same set $A = \{a_1, a_2, ..., a_n\}$ of *n objects*. Then we have the relation set $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_m\}$, where each of *m* rankings (preference relations) $\lambda = \{a_1 \succ a_2 \succ ... \sim a_s \sim a_t \succ ... \sim a_n\}$ may include $\succ$, a strict preference relation π, and $\sim$, an equivalence (or indifference) relation ν, so that $\lambda = \pi \cup \nu$. Such a relation λ is generally called a *weak order*. Thus, in this paper we will use the terms *ranking*, *preference relation* and *weak order* as synonyms. The relation set Λ can be titled a *preference profile* for the given *m* experts.

For example, let $n = 6$, $m = 5$, then we can have four following rankings of alternatives:

$$\begin{aligned}
&\lambda_1: a_1 \succ a_2 \succ a_6 \succ a_4 \succ a_3 \sim a_5,\\
&\lambda_2: a_4 \succ a_5 \succ a_1 \succ a_2 \succ a_3 \succ a_6,\\
&\lambda_3: a_2 \succ a_5 \succ a_1 \succ a_3 \succ a_4 \succ a_6,\\
&\lambda_4: a_6 \succ a_3 \succ a_4 \succ a_2 \succ a_1 \succ a_5,\\
&\lambda_5: a_3 \succ a_4 \succ a_2 \succ a_6 \succ a_5 \succ a_1.
\end{aligned} \tag{1}$$

Our aim is to determine a single preference relation that would give an integrative characterization of the alternatives. Let a space Π be a set of all $n!$ linear (strict) order relations $\succ$ on A. Each linear order corresponds to one of permutations of first n natural numbers $\mathbf{N_n}$. We will consider a permutation $\beta \in \Pi$ of the alternatives $a_1, ..., a_n$ to represent the preference profile Λ and will call it *consensus ranking*. It is desirable that, in some sense, β would be nearest to every of the rankings $\lambda_1, ..., \lambda_m$.

An example of the space of weak orders for $n = 4$ is shown in Fig. 1, where each node corresponds to one weak order. The space has 75 elements of which 24 are linear orders. The linear orders are shown in Fig. 2.

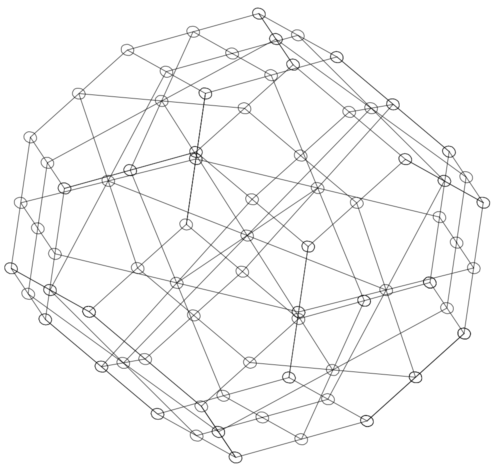

Fig. 1. Space of weak orders for $n = 4$.

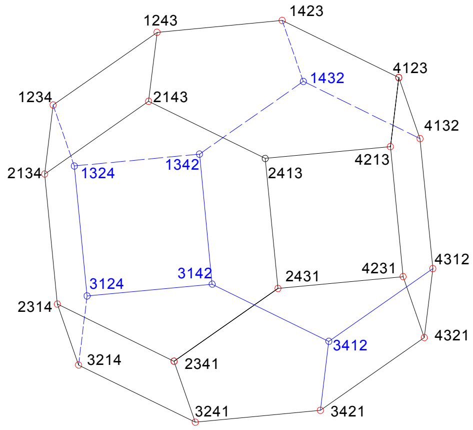

Fig. 2. Subspace Π of linear orders for $n = 4$.

It is clear that the problem described above is very similar to the problem of voting or group decision where A is a set of n alternatives or candidates which are ranked by a group of m individuals (voters, experts, focus groups, criteria, etc.).

The ranking λ can be represented by a $(n \times n)$ *relation matrix* $\mathbf{R} = [r_{ij}]$ whose rows and columns are labeled by the objects a and

$$r_{ij} = \begin{cases} 1 \text{ if } a_i \succ a_j \\ 0 \text{ if } a_i \sim a_j \\ -1 \text{ if } a_i \prec a_j \end{cases}. \tag{2}$$

The symmetric difference distance function $d(\lambda_k, \lambda_l)$ between two rankings λ_k and λ_l is defined by formula

$$d(\lambda_k, \lambda_l) = \sum_{i<j} | r_{ij}^k - r_{ij}^l | \tag{3}$$

and may be understood as the number of disagreements between two rankings. Here only elements of the upper triangle submatrix, r_{ij}, $i < j$, of matrix $\mathbf{R}$ are summed up. The value of $d(\lambda_1, \lambda_2)$ between the first two rankings of our example profile (1) is equal to 0+0+2+2+0+0+ +2+2+0+0+1+2+0+2+2 = 15.

The distance between arbitrary ranking λ and profile Λ can then be defined as follows:

$$D(\lambda, \Lambda) = \sum_{k=1}^{m} d(\lambda, \lambda_k) = \sum_{i<j} \sum_{k=1}^{m} \left| r_{ij}^k - r_{ij} \right|. \tag{4}$$

From (2), supposing $r_{ij} = 1$ for all $i < j$ that corresponds to the natural linear order $a_1 \succ a_2 \succ \ldots \succ a_n$, it is clear that for any $k = 1, \ldots, m$ we have $| r_{ij}^k - r_{ij} | = |1 - 1| = 0$ if $a_i^k \succ a_j^k$; $| r_{ij}^k - r_{ij} | = | 0 - 1 | = 1$ if $a_i^k \sim a_j^k$ and $| r_{ij}^k - r_{ij} | = | -1 - 1 | = 2$ if $a_i^k \prec a_j^k$. Thus, denoting $| r_{ij}^k - r_{ij} |$ through d_{ij}^k we have

$$D(\lambda, \Lambda) = \sum_{i<j} \sum_{k=1}^{m} d_{ij}^k , \tag{5}$$

where

$$d_{ij}^k = \begin{cases} 0 & \text{if } a_i^k \succ a_j^k \\ 1 & \text{if } a_i^k \sim a_j^k \\ 2 & \text{if } a_i^k \prec a_j^k \end{cases}. \tag{6}$$

We can now define a $(n \times n)$ *profile matrix* $\mathbf{P} = [p_{ij}]$ where

$$p_{ij} = \sum_{k=1}^{m} d_{ij}^k, \quad i, j = 1, \ldots, n, \tag{7}$$

and the number of voters m of the profile A is present in each of the matrix elements as $\frac{1}{2}(p_{ij} + p_{ji}) = m$, $i, j = 1, \ldots, n$. Thus, the value $0.5p_{ij}$ can be understood as the number of preferences a_j over a_i.

For the example profile (1), we have the following profile matrix $\mathbf{P}$:

$$[p_{ij}] = \begin{bmatrix} 0 & 6 & 4 & 6 & 6 & 4 \\ 4 & 0 & 4 & 6 & 2 & 2 \\ 6 & 6 & 0 & 4 & 5 & 4 \\ 4 & 4 & 6 & 0 & 2 & 4 \\ 4 & 8 & 5 & 8 & 0 & 6 \\ 6 & 8 & 6 & 6 & 4 & 0 \end{bmatrix}, \tag{8}$$

where, for instance, p_{24} = 0+2+0+2+2 = 6, the number of experts choosing a_4 over a_2 is 3 and the number of experts choosing a_2 over a_4 is 2.

2.2. Finding consensus ranking

How to find a single preference relation that would give an integrative characterization of the preference profile Λ described by matrix P? Condorcet in 1785, see [4], proposed a very natural and now well-known procedure of handling the paired-comparison data contained in the matrix P: in each comparison, the preferred object is the object preferred by a majority of voters, i.e. $a_i \succ a_j$ if and only if $p_{ij} > p_{ji}$. However, the binary relation defined by Condorcet's rule is not necessarily transitive, i.e. it can be that $a_i \succ a_j$ and $a_j \succ a_k$ while $a_k \succ a_i$. This Condorcet's Paradox of Voting may occur rather frequently: its chances are usually even greater than 50%.

Published in 1951 Arrow's Impossibility Theorem [5] has shown that no voting method can satisfy the following three desirable (natural) properties (axioms):

(P1) *unanimity* (if alternative a_i is ranked above a_j for all orderings λ_1, λ_2, ..., λ_m, then a_i is ranked higher than a_j by β),

(P2) *non-dictatorship* (there is no k-th voter whose preferences always prevail), and

(P3) *independence of irrelevant alternatives* (for two preference profiles Λ and Λ' such that for all k-th voters alternatives a_i and a_j have the same order in Λ and Λ', alternatives a_i and a_j have the same order in $\beta(\Lambda)$ and $\beta(\Lambda')$.

Thus, Arrow's theorem can serve as a thorough justification of Condorcet's paradox which means that a preference profile is not necessarily transitive even if each k-th ranking is a linear order.

In this situation, a reasonable way to get over the difficulty is to find such a linear order (permutation) $\beta \in \Pi$ of objects of A that the distance $D(\beta, \Lambda)$ from β to the profile Λ is minimal, that is

$$\beta = \arg\min_{\lambda \in \Pi} D(\lambda, \Lambda). \tag{9}$$

Thus, a solution of the optimization problem (9) is the consensus linear ranking β that is also called *median order*. Every permutation of objects of A corresponds to transposition of the profile matrix rows and columns. Hence, the problem (9) means the *determination of such a transposition of profile matrix rows and columns that the sum of elements of its upper triangle submatrix is minimal*. It should be noticed that the problem may have more than one optimal solution.

The space of solutions for this problem is great and this problem has been proven to be *NP*-hard [6]. However, for reasonable problem sizes (up to $n \approx 50$) there are exact algorithms for them to be applied, see, for example, [4, 7, 8]. They typically use the branch and bound (B&B) technique and, as a rule, are rather sophisticated and not easily realized in programming code. Trying to overcome the demerits, an algorithm has been proposed which is discussed in Section 3.

The application of median order as a consensus relation has been justified in [9] using an axiomatic characterization of the distance measure $d(\alpha_k, \alpha_l)$. More profound justification, taking into account conditions of Arrow's theorem, has been done in [10].

3. PROCEDURE FOR CONSENSUS RELATION DETERMINATION

The input of the algorithm will be the profile matrix $\mathbf{P} = [p_{ij}]$ and the output will be the optimal transposition β, and the corresponding upper bound value of the distance function $l^u = D(\beta, \Lambda)$. A transposition of rows and columns of matrix $\mathbf{P}$ is to be represented by a permutation of first n natural numbers $\mathbf{N_n} = \{1, 2, ..., n\}$.

The algorithm of determining an optimal transposition of rows and columns of the profile matrix uses the *recursive B&B technique* [11] which turned out to be suitable for the case, and works as described below.

3.1. Least distance

Matrix $\mathbf{P}$ can be characterized by a *least distance* D_{least} from its preference profile Λ to some linear order. A lesser element of each pair (p_{ij}, p_{ji}) is included into the distance, that is

$$D_{least} = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \min(p_{ij}, p_{ji}) = \sum_{i<j} \min(p_{ij}, p_{ji}), \tag{10}$$

where $\min(a,b) = \begin{cases} a \text{ if } a < b \\ b \text{ if } b < a \end{cases}.$

It is clear that if matrix $\mathbf{P}$ is transitive, i.e. $p_{ik} \le p_{ki}$ if $p_{ij} \le p_{ji}$ and $p_{jk} \le p_{kj}$, $i \ne j \ne k = 1, ..., n$ (this means all initial rankings are consistent), then $D_{least} = D(\beta, \Lambda)$ and D_{least} is an accessible value. An inverse proposition is also valid. Additionally evident that D_{least} does not depend on a transposition of profile matrix $\mathbf{P}$ rows and columns.

In view of Condorcet's Paradox the equality $D_{least} = D(\beta, \Lambda)$ is not always satisfied and its validity can be established only after determination of *β*. If matrix **P** is intransitive, then $D(\beta, \Lambda) > D_{least}$ and D_{least} becomes inaccessible.

3.2. Search tree

The algorithm investigates a tree-structured solutions space (see Fig. 3). Each node of the solution tree is in one-to-one correspondence with a set $S = \{s_1, s_2, \ldots, s_k\}$, which is considered to be a representative (or leader) of all solutions containing it as a leading part. The tree root is leader of absolutely all feasible solutions and for it $S = \varnothing$. At the next (first) tree level there are n leaders of cardinality 1. Each of the leaders has $n - 1$ successors of cardinality 2 at the second level. Thus, each of k-th level leaders has $n - k$ successors of cardinality $|S| = k + 1$, $k = 0, \ldots, n - 1$. If $|S| < n - 1$ then an appropriate solution S is to be called current *partial solution*. Given $|S| = n - 1$ the set S is to be a *complete solution* as, in this case, it defines an order of all elements of A.

A leader is built up of elements of $\mathbf{N_n} = \{1, 2, ..., n\}$, hence, for any leader S there exists its complement $\bar{S} = T = \{t_1, t_2, \ldots, t_{k-n}\} = \mathbf{N_n} \setminus S$ containing elements of $\mathbf{N_n}$ which are not included into S. Notice that elements in T are always arranged in a lexicographical order.

Each leader S is built up by means of concatenation of its predecessor and first in order element t_l of T, i.e. $S = \{s_1, \ldots, s_{k-1}, s_k = t_l\}$, and at the same time t_l is removed from T. For example, let $n = 6$ then if at the search tree level $k = 3$ the leader $S = \{2, 1, 3\}$ and $T = \{4, 5, 6\}$ then at the next, fourth, level $S = \{2, 1, 3, 4\}$ and $T = \{5, 6\}$.

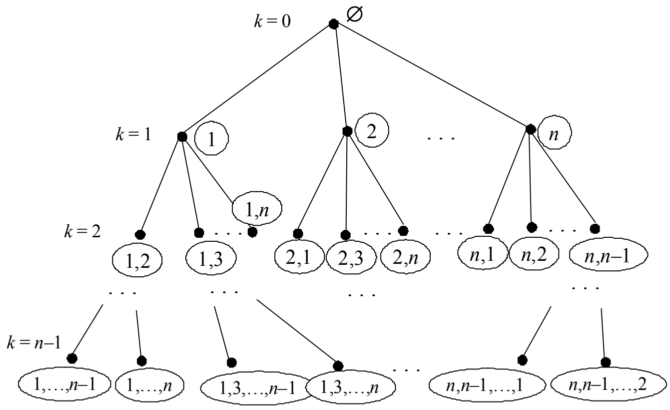

Fig. 3. Search tree.

3.3. Leader promise check

Each leader (partial solution) has an appropriate estimate D_{low} of a distance from profile Λ to the optimal linear order *β*. Let us call this a *low bound* D_{low}. The minimal value of a distance function for complete solutions generated to the moment is termed an *upper bound* D_u.

Leader S strictly defines a position of the matrix **P** rows and columns with indexes $s_1, \ldots, s_{k-1}$. Corresponding sum of elements of upper triangle submatrix of the matrix **P** is

$$D = \sum_{\substack{i=1,\ldots,k-1 \\ j=i+1,\ldots,k-1}} p_{s_i s_j} + \sum_{i,j=1,\ldots,n-k} p_{t_i t_j} . \tag{11}$$

Taking into account an expansion of the leader due to concatenation with element t_l gives

$$D_e = D + \sum_{i=1,\ldots,n-k} p_{s_k t_i}\,. \tag{12}$$

Finally, for the rest of matrix defined by elements of T, we can use the same principle as for determination of the least distance (10). Then we have

$$D_{low} = D_e + \sum_{\substack{i=1,\ldots,n-k \\ j=i+1,\ldots,n-k}} \min(p_{t_i t_j}, p_{t_j t_i})\,. \tag{13}$$

In the example in Fig. 4 $D = 51$, $D_e = 51 + 6 = 57$, $D_{low} = 57 + 4 = 61$.

A leader is considered to be promising in case the condition $D_{low} < D_u$ is satisfied. If $D_{low} \geq D_u$ (notice that due to (13) D_{low} is the least value for the given leader S) then it is clear that all solutions including this leader are hopeless, i.e. cannot be optimal.

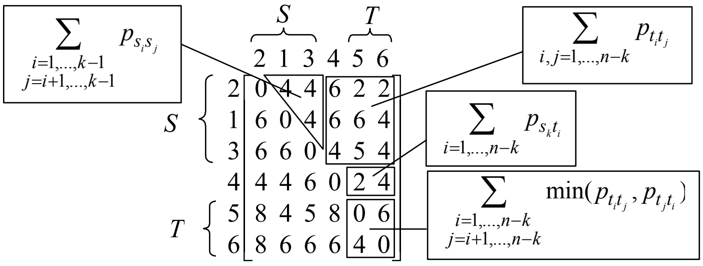

Fig. 4. Towards calculation of a low bound.

3.4. The algorithm

The algorithm (see Fig. 5) contains only two stages:

- *initialization*, where for the given matrix **P** the parameter D_{least} is calculated and all necessary variables acquire initial values, and
- *call of recursive procedure LEADER* (k, D) which accepts two parameters, k and D, where k is a number of the search tree level and D is a low bound component defined by a fixed order of leader elements as in formula (11). Initial values of the parameters are as follows: $k = 1$ и $D = 0$.

Procedure *LEADER* (k, D) contains the main cycle by l, where l is a leader number at k-th level. At each cycle step the current partial solution in form of leader S is generated. A new leader S every time defines new position of the matrix P rows and columns. For it D_e and D_{low} are calculated. If $D_{low} < D_u$ and $k < n - 1$ then the procedure $LEADER(k+1,\ D_e)$ is called in order to check the next level of search tree. It this way *branching* is realized. If $D_{low} < D_u$ and the solution is complete then it is memorized as a pair $\beta = S$ and $D_u = D_{low}$. If $D_{low} \geq D_u$ then the corresponding leader and all its successors are considered to be hopeless and they are *pruned*. Then we continue to see if any other incomplete solutions might feasibly lead to a better complete solution.

The search is continued until all hopeless solutions will be pruned. The algorithm is an exact one as it checks all the feasible incomplete solutions.

Solutions of the algorithm for our example profile (1) are reduced to Table 1. For this particular problem we have $\beta = \{a_4 \succ a_2 \succ a_1 \succ a_3 \succ a_6 \succ a_5\}$, $D_{least} = 55$, $D_u = 59$, i.e. the initial preference rankings are inconsistent and the profile matrix is non-transitive. The number of partial solutions (nodes of the search tree) for this example is 71.

$$D_{least} \leftarrow \sum_{\substack{i=1,\ldots,n\\ j=i+1,\ldots,n}} \min(p_{ij}, p_{ji});\ D_u \leftarrow \infty;\ S \leftarrow \varnothing;\ T \leftarrow \mathbf{N_n};\ \text{[initialization]}$$

$$LEADER\ (1, 0);\ \text{[call of the recursive procedure]}$$

$$\textbf{procedure}\ LEADER\ (k, D):\ \text{[the recursive procedure definition:]}$$

$$\textbf{for}\ l = 1\ \textbf{to}\ n-k+1\ \textbf{do if}\ D_u \neq D_{least}\ \textbf{then}$$

$$\begin{cases} s_k \leftarrow t_l;\ T \leftarrow T - \{t_l\};\ \text{[branching]} \\ D_e \leftarrow D + \sum\limits_{i=1,\ldots,n-k} p_{s_k t_i};\ \text{[change of distance due to expanding the leader } S\text{]} \\ D_{low} \leftarrow D_e + \sum\limits_{\substack{i=1,\ldots,n-k\\ j=i+1,\ldots,n-k}} \min(p_{t_i t_j}, p_{t_j t_i});\ \text{[modification of the lower bound]} \\ \textbf{if}\ D_{low} < D_u\ \textbf{then} \begin{cases} \textbf{if}\ k < n-1\ \textbf{then}\ LEADER(k+1, D_e) \\ \qquad \textbf{else} \begin{cases} \beta \leftarrow S;\ \text{[keeping the complete solution]} \\ D_u \leftarrow D_{low}; \end{cases} \\ T \leftarrow T \cup \{s_k\};\ \ S \leftarrow S - \{s_k\};\ \ \text{[pruning]} \end{cases} \end{cases}$$

Fig. 5. Algorithm for consensus relation determination.

4. CONSENSUS RELATION UNCERTAINTY ASSESSMENT

Now let us take into account the fact that there always exists some uncertainty in matrix **P** definition as object rankings may be erroneous by different reasons, both subjective and objective. This section is mainly based on ideas from [12].

4.1. Radius of stability

Let $\mathbf{P} \in \mathbf{Z^n}$ and in the space $\mathbf{Z^n}$ a norm is defined.

If $p_{11}, \ldots, p_{nn}$ are given with uncertainty not exceeding ε, and uncertainties in elements definition are independent, then by decision of the problem over **P** we would like to believe that it is solved over any matrix **Q** belonging to a sphere $S_\varepsilon(P)$ of a radius ε with center in **P**, that is

$$S_\varepsilon(P) = \{Q \mid Q \in \mathbf{Z^n}, \|Q\| < \varepsilon\}, \tag{14}$$

where $\|Q\|$ is the norm of **Q**.

This belief is based on an assumption that the solution is correct. However, in real situation

- p_{ij} are always given with limited accuracy and
- in fact, there exists some particular $\mathbf{P'}$, about which it is only known that $\mathbf{P'} \in S_\varepsilon(P)$.

It should be noticed that $\mathbf{P'}$ by no means always coincides with **P**. Indeed, for any $\varepsilon > 0$ one can give an example (see section 3) of **P** such that for some $\mathbf{P'} \in S_\varepsilon(P)$ sets of optimal solutions on **P** and on $\mathbf{P'}$ are not intersected.

In this situation, having solved the problem on **P**, we will know nothing about the problem solution on really existing matrix $\mathbf{P'}$. To resolve the challenge, it seems to be reasonable to have an algorithm that by **P** gives out $\rho(P)$ (which will be called *radius of stability*) such that the problem solution on **P** is also a solution on any $\mathbf{P'} \in S_\rho(P)$. Thus, $\rho(P)$ defines the maximal admissible error in assigning numerical problem parameters. Then the problem can be considered to be correctly solved under the condition $\varepsilon < \rho(P)$, and, otherwise, the initial numerical data need to be defined more exactly. The problem solution without the revision would be meaningless.

Let $B(P)$ be a set of indexes of optimal solutions of some problem on **P**. Denote the problem through Z_P. If $B(P)$ includes all feasible solutions, then we suppose, by definition, that $\rho(P) = 0$. Otherwise, it can be easily shown that $\rho(P) > 0$.

Table 1. Solutions of the algorithm.

Step	S	T	D_{low}	$\gtreqless D_u$	Decisions and comments
0	{ }	1,2,3,4,5,6			$D_u = \infty$, $D_{least} = 55$
1	1	2,3,4,5,6	61	<	branching
2	1,2	3,4,5,6	63	<	branching
3	1,2,3	4,5,6	63	<	branching
4	1,2,3,4	5,6	63	<	branching
5	1,2,3,4,5	6	65	<	the complete solution is {1,2,3,4,5,6}, $D_u = 65$
6	1,2,3,4,6	5	63	<	the complete solution is {1,2,3,4,6,5}, $D_u = 63$
7	1,2,3,5	4,6	71	>	pruning
8	1,2,3,6	4,5	65	>	pruning
9	1,2,4	3,5,6	65	>	pruning
10	1,2,5	3,4,6	71	>	pruning
11	1,2,6	3,4,5	67	>	pruning
12	1,3	2,4,5,6	63	=	pruning
13	1,4	2,3,5,6	63	=	pruning
14	1,5	2,3,4,6	75	>	pruning
15	1,6	2,3,4,5	71	>	pruning
16	2	1,3,4,5,6	57	<	branching
17	2,1	3,4,5,6	61	<	branching
18	2,1,3	4,5,6	61	<	branching
19	2,1,3,4	5,6	61	<	branching
20	2,1,3,4,5	6	63	=	branching
21	2,1,3,4,6	5	61	<	the complete solution is {2,1,3,4,6,5}, $D_u = 61$
22	2,1,3,5	4,6	69	>	pruning
...	…	…	…	…	…
60	4,2,1,3,5	6	61	=	pruning
61	4,2,1,3,6	5	59	<	the complete solution is {4,2,1,3,6,5}, $D_u = 59$
62	4,2,1,5	3,6	61	>	pruning
63	4,2,1,6	3,5	61	>	pruning
64	4,2,3	1,5,6	59	=	pruning
65	4,2,5	1,3,6	59	=	pruning
66	4,2,6	1,3,5	61	>	pruning
67	4,3	1,2,5,6	61	>	pruning
68	4,5	1,2,3,6	65	>	pruning
69	4,6	1,2,3,5	67	>	pruning
70	5	1,2,3,4,6	69	>	pruning
71	6	1,2,3,4,5	67	>	pruning; no unchecked uncomplete solutions exist, the last complete solution is optimal

Let as consider transition from **P** to **P** ⊕ **Q**, where **Q** < ε and ⊕ is the operation of combining matrices **P** and **Q**. If at any such transition no non-optimal on **P** solution becomes optimal on **P** ⊕ **Q** (i.e. the gap between optimal and non-optimal solutions remains), then **P** is ε-stable. This condition can be written as follows:

$$B(\mathbf{P} \oplus \mathbf{Q}) \subseteq B(\mathbf{P}). \tag{15}$$

Now let $\rho(P) = \sup \varepsilon$ where supremum is taken by all $\varepsilon > 0$, for which Z_P is stable.

Under $\rho(P) = 0$ the sphere $S_\rho(P)$ degenerates to a point. Otherwise, solution of all Z_Q under $Q \in S_\rho(P)$ in view of (15) necessarily is among solutions of the problem Z_P.

4.2. Examples

In this section we illustrate the above statements with examples. All of the examples are produced for case $n = 4$. This value of n allows to demonstrate meaningful instances while still keeping a satisfactory level of obviousness. The corresponding space of 75 weak orders has been shown in Fig. 1 and its 2D representation is shown in Fig. 6. Every vertex in this diagram corresponds to one possible ranking (they are represented by indexes i of objects a_i, and strict order symbols $\succ$ are omitted, i.e. $\{1234\} \equiv \{1\succ2\succ3\succ4\} \equiv \{a_1\succ a_2\succ a_3\succ a_4\}$ or $\{1\sim324\} \equiv \{1\sim3\succ2\succ4\} \equiv \{a_1\sim a_3\succ a_2\succ a_4\}$, and so on). Strict orderings in this space forms the solution space Π of the problem (9). The space is closed but in order to have the possibility to represent it on a plane we break some vertices into two with the same designation (a copy of each of the elements is white and elements of all corresponding pairs are connected with dashed line). Each edge has a number indicating the distance $d(\lambda_k, \lambda_l)$ between corresponding two rankings. The central vertex of each hexagon in the space is connected to the element $\{1\sim2\sim3\sim4\}$ with the distance equal to 3.

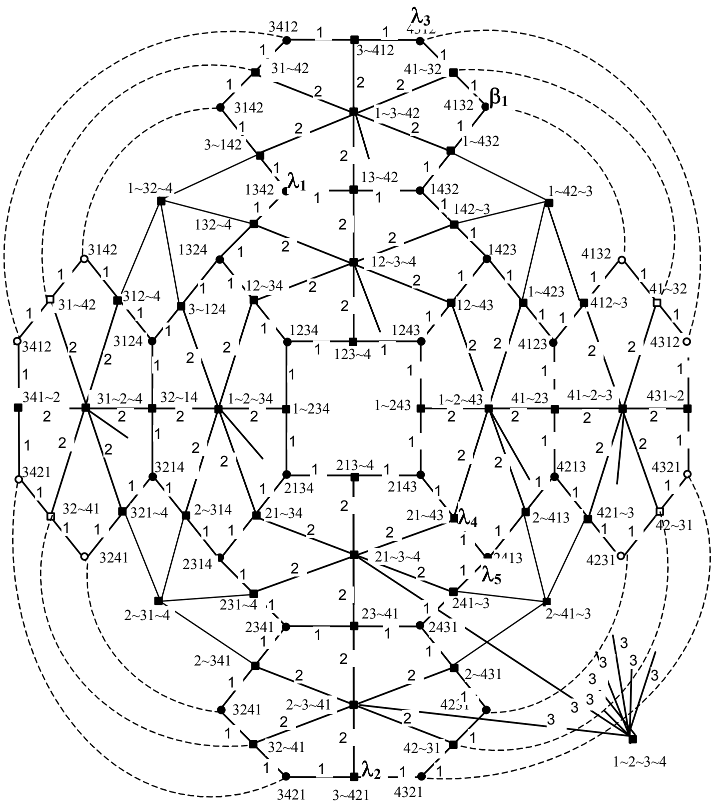

Fig. 6. Two-dimensional representation of the space of all possible weak orders for $n = 4$ (compare with Fig. 1).

Example of stable solution. Let a preference profile be given as follows: λ_1: 1342; λ_2: 3~421; λ_3: 4312; λ_4: 21~43; λ_5: 2413 (see Fig. 6). The profile matrix is

$$[p_{ij}^1] = \begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 6 & 6 \\ 6 & 4 & 0 & 7 \\ 3 & 4 & 3 & 0 \end{bmatrix}.$$

The B&B algorithm gives the solution β_1: 4132, $D(\beta_1, \Lambda_1) = 24$.

Now we change the object order in the ranking λ_4. Let it be 12~43. In this case the profile matrix is

$$[p_{ij}^2] = \begin{bmatrix} 0 & 4 & 4 & 6 \\ 6 & 0 & 6 & 7 \\ 6 & 4 & 0 & 7 \\ 4 & 3 & 3 & 0 \end{bmatrix}.$$

The solution β_2 is the same: 4132, but $D(\beta_2, \Lambda_2) = 22$. It means that, in this case, the profile matrix allows to have some uncertainty in its elements.

Examples of non-intersected solutions. The solutions for individual problems are reduced in Table 2. The initial preference profile for all of them is Λ_1. An individual problem is obtained by a small change in one of the five rankings; they are in the first column of Table 2.

It can be seen from the examples that sets of optimal solutions on **P** and on P_3, P_4, …, P_7 are not intersected.

Table 2. Examples of non-intersected solutions.

Ranking changed	λ_1: 3142	λ_2: 1~324	λ_3: 3412	λ_2: 2~341	λ_1: 3412
Profile matrix	$\begin{bmatrix} 0 & 6 & 6 & 7 \\ 4 & 0 & 6 & 6 \\ 4 & 4 & 0 & 7 \\ 3 & 4 & 3 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 4 & 3 & 5 \\ 6 & 0 & 6 & 6 \\ 7 & 4 & 0 & 6 \\ 5 & 4 & 4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 6 & 6 \\ 6 & 4 & 0 & 5 \\ 3 & 4 & 5 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 4 & 7 \\ 4 & 0 & 5 & 6 \\ 6 & 5 & 0 & 6 \\ 3 & 6 & 4 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 6 & 6 & 9 \\ 4 & 0 & 6 & 6 \\ 4 & 4 & 0 & 7 \\ 1 & 4 & 3 & 0 \end{bmatrix}$
Optimal solution	β_3: 4321 $D(\beta_3, \Lambda_3) = 22$	β_4: 1432 $D(\beta_4, \Lambda_4) = 24$	β_5: 3421 $D(\beta_5, \Lambda_5) = 26$	β_6: 2413 $D(\beta_6, \Lambda_6) = 24$	β_7: 4312 $D(\beta_7, \Lambda_7) = 22$

Let us fix Λ (and, consequently, **P**) and consider the matrix $\mathbf{X} \in \mathbf{Z}^{\mathbf{n}}$. Denote $D(\beta_k, \Lambda)$ through $D_k(P)$. For an arbitrary pair of solutions β_k and β_l, $D_k(P) < D_l(P)$, one can state the following problem:

$$||X|| \rightarrow \min, D_k(P \oplus X) \geq D_l(P \oplus X). \tag{16}$$

Consideration of the problem (16) can allow to obtain a particular formula for the radius of stability and this is a direction of future investigations on the topic. The approach described would allow to assign a definite level of uncertainty to ordinal measurement result. This situation is typical for quantitative physical measurements. The similar possibility implemented for ordinal measurement would considerably increase the level of their reliability.

5. CONCLUSION

One can argue that initial rankings are subjective as they are obtained without the use of a measuring instrument. The possible answer to this point may be justified with the help of a substitution of m individuals by m sensors in our problem description (see section 2.1). Then the problem of consensus ranking determination turns into one of *sensor data fusion* problems. An interesting example concerning this is given in [13] which illustrates a consensus ranking characterizing the level of threat of n targets indicated on the radar screen of a fighter aircraft. There are analysed data from a set of m sensors measuring the size, shape and speed of approach and range to the target. In this case, the sensors (as measuring instruments) give out objective information at the same time voting in the same manner as subjective people. Here the profile matrix describing the sensor "opinions" seems to have no distinction from one produced by individuals. And Condorcet's paradox should be getting over as earlier, that is by means of solving the problem (9). Thus, the problem is a subject invariant.

Another objection can be that the rankings are non-empirical as they are obtained of a thought experiment and reflect unobservable relations. To comment this position let us remember how reliable are our assumptions when measuring in ratio scale. We believe that the attribute we measure is directly connected to a property under investigation, we think that measurement errors are distributed in accordance to a known law, we rely on SI units and so on. However, our security is illusory because the measured attribute can be misleading, the error distribution is completely out of our pre-diction, SI units partition into fundamental and derived ones is only a convention and the measurement instrument is not calibrated as believed, etc. Thus the level of our confidence to classical measurement is arguable. So do rankings, clearly to a more considerable extent.

Finally our conclusion is that consensus rankings as they were described above can be treated as ordinal scale measurement results with a wide area of application in practical metrology , quality management and control.

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