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STUDY OF RESULTS OF COMPARISON OF SELECTED UNCERTAINTIES

The paper presents some situations in which it is necessary to compare uncertainties. Several situations have been distinguished, in which the comparison result may decide about the following factors: the selection of the approximated method of evaluation of expanded uncertainty, the dominant position of standard component uncertainties, and the planned number of measurements. The paper present the differences in evaluation, caused by selecting various uncertainties, whose relationships are examined. The accuracy of obtained evaluation is used as a criterion for selecting uncertainties to be compared.

1. INTRODUCTION

In many situations concerning the evaluation of uncertainty of measurement results, it is necessary to compare the values of uncertainty in order to choose correctly the approximated method of evaluation. Such comparisons are also performed in other situations, for example when we examine the dominating component of the standard uncertainty, or when we are planning the number of observations. In the following chapters we will analyse these three situations.

So far, uncertainties whose relations are to be examined have been chosen on the basis of the comparison of the limit values of errors. Limit error results from imperfection of the measuring instrument, described by its accuracy index. It is equivalent to examining the relations between the limit values of uncertainties. However in view of the new methods of uncertainty evaluation, it seems to be more adequate to use the relations between standard uncertainties. Standard uncertainties are components of a vector sum which expresses the combined standard uncertainty. At the same time they are characteristic parameters of components of probability distribution. The aim of this paper is to examine the effects and differences in evaluation of the expanded uncertainty, caused by selecting different uncertainties, whose respective relations are examined.

We will consider the case of simple direct measurement with two standard uncertainties, where one of them is type A and other is type B standard uncertainty, and the component errors are independent variables. In this case the combined standard uncertainty u_c and the expanded uncertainty u_e will be equal:

$$u_c=\sqrt{u_A^2+u_B^2} \tag{1}$$

$$u_e = k(\alpha) \cdot u_c = k_{NR}(\alpha) \tag{2}$$

The combined standard uncertainty is the standard deviation of the convolution of component distributions for assumed probability α. If we assume that random errors have the normal distribution, and type B uncertainty is caused by the measuring device whose errors have the rectangular distribution, then we obtain a convolution of the normal and the rectangular distribution $N*R$ [1, 4]. The coverage factor $k(\alpha)$, is in this case a standardised variable of the convolution of these two distributions and will be denoted by $k_{NR}(\alpha)$ in opposition to the symbol $k_N(\alpha)$ for the normal distribution and $k_R(\alpha)$ for the rectangular distribution.

It can be shown that $k_{NR}(\alpha)$ is not only a function of probability α, but also a function of the ratio of the standard deviations σ_N of normal distribution (or its estimator $\bar{S}$) and σ_R of the rectangular distribution, or of the rate of type A and type B standard uncertainties [3, 4, 6].

$$k_{NR}(\alpha) = f\left(\alpha, \frac{\sigma_N}{\sigma_R}\right) \tag{3}$$

The above simple case of direct measurement involves certain problem in evaluating expanded uncertainty, since the central limit theorem about the convergence of a convolution of component distributions to the normal distribution is not applicable here, because of the limited number of components.

2. METHODS OF EVALUATION OF EXPANDED UNCERTAINTY

2.1. Description of methods

For further analysis we assume that the method based on the convolution of the normal and the rectangular distributions is an accurate way of evaluating expanded uncertainty [3, 4, 5, 6]. It is also assumed that approximated methods are based on the hypothesis that unknown convolution can be approximated by the distribution of the component of bigger uncertainty. The above hypothesis is valid for an extreme cases: if $\sigma_N/\delta_R \to \infty$, than $k(\alpha) \to k_N(\alpha)$ and if $\sigma_R/\sigma_N \to \infty$ than $k(\alpha) \to k_R(\alpha)$. However this hypothesis does not define the possibilities of estimation if the standard deviations are related to each other is equal or near equal.

Using the value of $k_{NR}(\alpha)$, for a known convolution of distributions $N*R$ [1, 2, 4, 5], we assume that the measure of accuracy of $k(\alpha)$ is given by the error:

$$\delta_k = \frac{|k(\alpha) - k_{NR}(\alpha)|}{k_{NR}(\alpha)} \cdot 100\% \tag{4}$$

The accuracy of evaluation of coverage factor $k(\alpha)$ decides about the accuracy of the approximated method. We assume a particular value of error, equal to 20%. When the error exceeds this value, the analysed approximated method should be rejected as not sufficiently accurate. This assumption may be verified for different measurement situations.

The values of the errors δ_1 and δ_2 caused by selecting various uncertainties whose relationships are examined will be show on the figures presented below.
Where:

$$\delta_1 = \frac{|\, k_N(\alpha) - k_{NR}(\alpha)\,|}{k_{NR}(\alpha)} \cdot 100\% \tag{5}$$

$$\delta_2 = \frac{|\, k_R(\alpha) - k_{NR}(\alpha)\,|}{k_{NR}(\alpha)} \cdot 100\% \tag{6}$$

In this case the coverage factor $k(\alpha)$, assumed the values of the standardised variables $k_N(\alpha)$ of normal distribution or $k_R(\alpha)$ of rectangular distribution respectively. Where:

$$k_R(\alpha) = \sqrt{3} \cdot \alpha \quad \text{and} \quad k_N(\alpha) = z(\alpha) \tag{7}$$

2.2. Relations between standard deviations

The results of examining the accuracy of evaluation of expanded uncertainty when the relations between standard deviation σ_N of normal distribution, and standard deviation σ_R of rectangular distribution are considered is known [3, 5, 6]. Fig. 1 shows the absolute values of errors δ_1 and δ_2 for the assumed probability $\alpha = 0.99$.

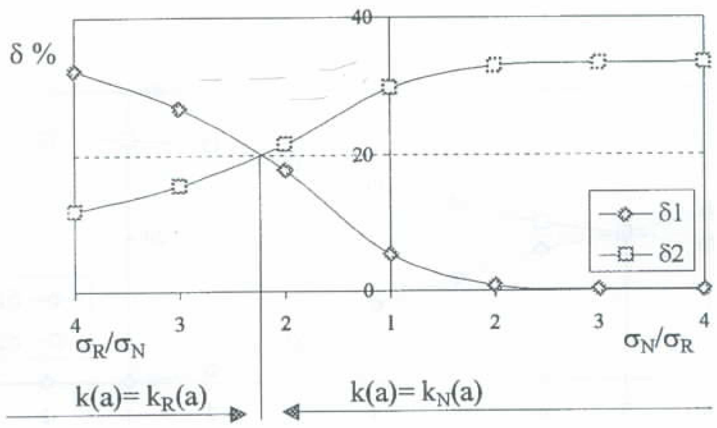

Fig. 1. Absolute values of error of the coverage factor $k(\alpha)$, estimated on the basis of relation between standard deviations σ_N and σ_R, for $\alpha = 0.99$

The values of errors do not exceed the assumed value 20%, if the relations (8) and (9) are satisfied.

$$u_e = k(\alpha) \cdot u_c = k_{NR}(\alpha) \tag{2}$$

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The accuracy of evaluation of coverage factor $k(\alpha)$ decides about the accuracy of the approximated method. We assume a particular value of error, equal to 20%. When

$$\frac{u_A}{u_B}=\sqrt{\frac{1}{n}}\cdot\frac{\sigma_N}{\sigma_R}=\sqrt{\frac{1}{n}}\cdot\frac{\overline{S}}{\sigma_R} \tag{10}$$

As follows from Fig. 2 the error of the factor $k(\alpha)$ does not exceed the assumed value of 20% under condition that determined relations are satisfied. These relations are different for different number of degree of freedom n. An increase of n is followed by an increase of the interval in which factor $k(\alpha)$ assumes the values of the standardised variable $k_N(\alpha)$ of the normal distribution.

2.3. Relations between limiting values of expanded uncertainties

If we assume that type B expanded uncertainty has a rectangular distribution, then this uncertainty is expressed by relation (12), and its limit value for $\alpha=1$ is equal to the limit error Δ_l of a measuring device (11).

$$u_{\mathrm{Bel}}=\Delta_l \tag{11}$$

$$u_{\mathrm{Be}}=k_R(\alpha)\cdot u_{\mathrm{B}}=\sqrt{3}\cdot\alpha\cdot\frac{\Delta_l}{\sqrt{3}}=\alpha\cdot\Delta_l \tag{12}$$

Therefore we will examine the relation between the limit values of the error of the measuring device and the limit values of type A expanded uncertainties. These relations may be defined in various ways, because $k(\alpha)$ may assume the values of the standardised variable $z(\alpha)$ for the normal distribution, or of the standardised variable $t_{q,m}$, for the t Student distribution. Moreover, because the error of the measuring device affects each result of the sample, the value of the compared quantities can be assessed by the limit value of expanded uncertainty of a single measurement. If we present these three different evaluations as functions of the estimator $\overline{S}$ of the standard deviation σ_N, we will get the following evaluations for uncertainties:

– for a single result

$$u_{\mathrm{Acl1}}=z(\alpha)\cdot\overline{S}=3\cdot\overline{S} \tag{13}$$

– for the mean value

$$u_{\mathrm{Acl2}}=z(\alpha)\cdot\frac{\overline{S}}{\sqrt{n}}=3\cdot\frac{\overline{S}}{\sqrt{n}} \tag{14}$$

$$u_{\mathrm{Acl3}}=t_{q,m}\cdot\frac{\overline{S}}{\sqrt{n}} \tag{15}$$

where: $q=1-\alpha$ and $m=n-1$.

We want to compare the accuracy of evaluations which use the relations between different limit expanded uncertainties and of the evalutions assumed to be accurate.

In order to do so, we have to present the rates of type A and type B limit expanded uncertainties as a function of standard deviation σ_N and σ_R. We obtain the following relations:

$$\frac{u_{Ael1}}{u_{Bel}}=\sqrt{3}\cdot\frac{\sigma_N}{\sigma_R} \tag{16}$$

$$\frac{u_{Ael2}}{u_{Bel}}=\sqrt{\frac{3}{n}}\cdot\frac{\sigma_N}{\sigma_R} \tag{17}$$

$$\frac{u_{Ael3}}{u_{Bel}}=\frac{t_{q,m}}{\sqrt{3\cdot n}}\cdot\frac{\sigma_N}{\sigma_R} \tag{18}$$

We can see that the above relations are different for a different number of degrees of freedom n, except for the relation (16). It also follows from this relations that the ratio of the standard deviation σ_N/σ_R is always smaller than the ratio of limit expanded uncertainties. The results of the analysis are presented.

Fig. 3 shows the absolute values of error of the factor $k(\alpha)$, estimated on the basis of relation between limit values of expanded uncertainties u_{Ael1} for a single result, and u_{Bel}, for $\alpha=0.99$. The ratio (16) of these uncertainties is not a function of the number of degree of freedom.

As follows from Fig. 3 the error of the factor $k(\alpha)$ does not exceed the assumed value of 20% on condition of fulfilling the determined relations:

$$\text{If} \quad u_{Ael1} \geqslant u_{Bel} \quad \text{then} \quad k(\alpha)=k_N(\alpha) \tag{19}$$

$$\text{If} \quad u_{Bel} > u_{Ael1} \quad \text{then} \quad k(\alpha)=k_R(\alpha) \tag{20}$$

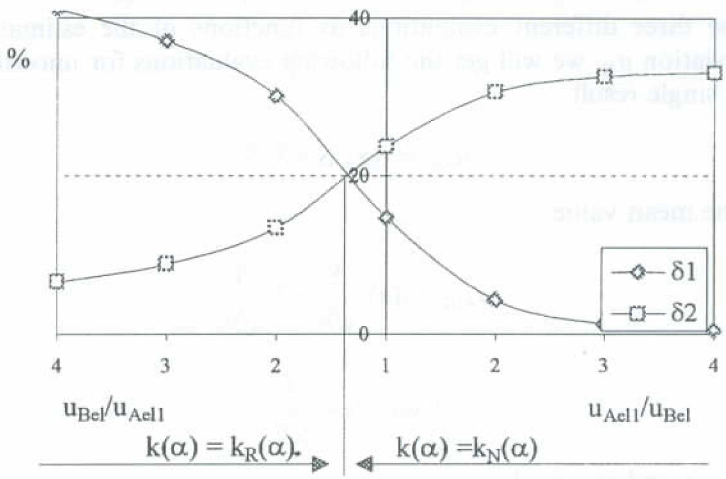

Fig. 3. Absolute values of error of the factor $k(\alpha)$, estimated on the basis of relation between limit expanded uncertainties u_{Ael1} for a single result and u_{Bel}, for $\alpha=0.99$

Figures 4a, 4b, 5a and 5b show the absolute values of errors δ_1 and δ_2, which result from the evaluation of the coverage factor $k(\alpha)$, when we examine the relationships between limit expanded uncertainties described by relations (17) and (18). These errors are presented for two different values of the degree of freedom: $n=5$ and $n=10$, and for assumed probability $\alpha=0.99$.

As follows from Fig. 4, the examination of relationship between limit value of expanded uncertainty of u_{Ael2} and limit value of expanded uncertainty u_{Bel} allows us to evaluate the value of the factor $k(\alpha)$ with an error not exceeding the assumed value of 20% on condition of fulfilling the determined relations. These relations are also different for different number of degree of freedom $\boldsymbol{n}$. An increase of $\boldsymbol{n}$ is followed by

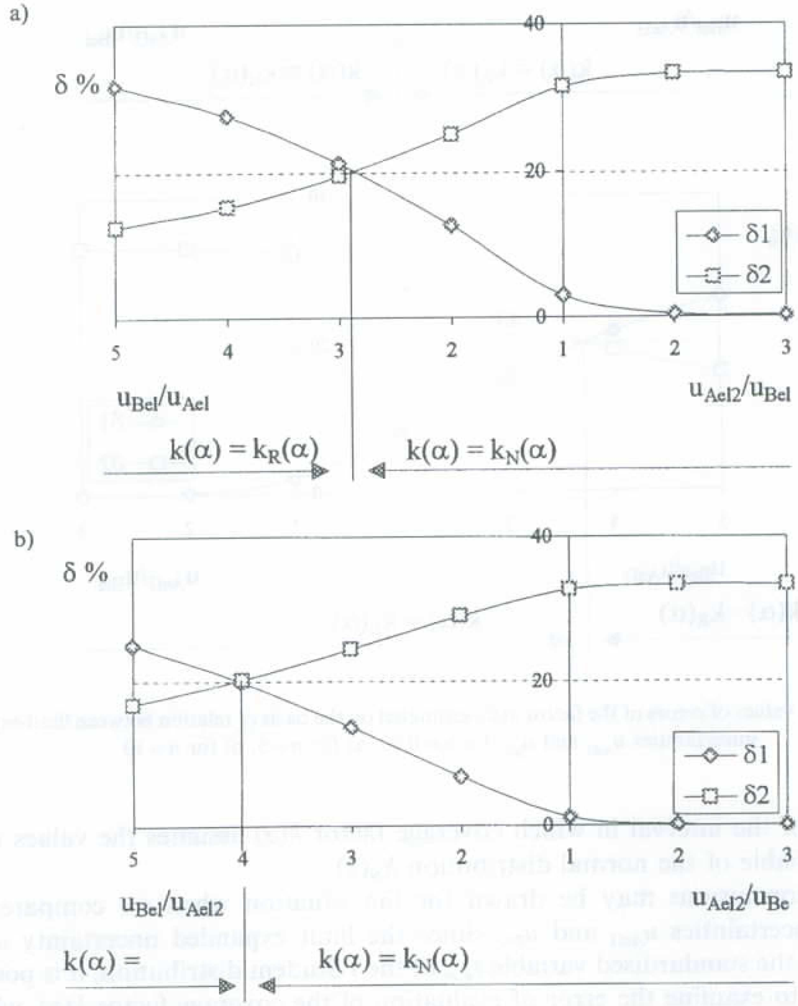

Fig. 4. Absolute values of errors of the factor $k(\alpha)$, estimated on the basis of relation between limit expanded uncertainties u_{Ael2} and u_{Bel}, for $\alpha=0.99$, a) for $n=5$, b) for $n=10$

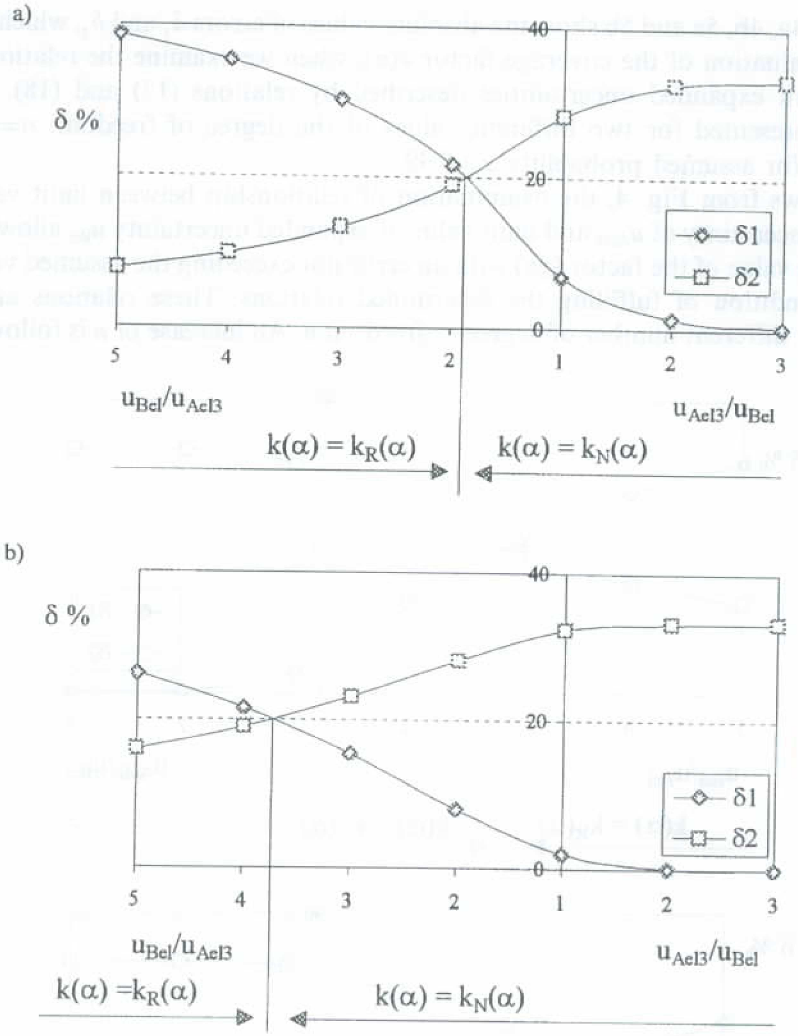

Fig. 5. Absolute values of errors of the factor $k(\alpha)$, estimated on the basis of relation between limit expanded uncertainties u_{Ael3} and u_{Bel}, for $\alpha=0.99$, a) for $n=5$, b) for $n=10$

an increase of the interval in which coverage factor $k(\alpha)$ assumes the values of the standard variable of the normal distribution $k_N(\alpha)$.

Similar conclusions may be drawn for the situation when we compare limit expanded uncertainties u_{Ael3} and u_{Bel}. Since the limit expanded uncertainty u_{Ael3} is expressed by the standardised variable $t_{q,m}$ of the t Student distribution, it is possible, additionally to examine the error of evaluation of the coverage factor $k(\alpha)$, when it takes the values of $t_{q,m}$. However in this situation the basis for accurate evaluation would be the convolution of t Student distribution and rectangular distribution.

The conclusion from the presented analysis seems to be ambiguous from the point of view of achieved accuracy of evaluation of coverage factor $k(\alpha)$. In all discussed situations, in which we examined the relations between the values of standard uncertainties, and between various values of limit expanded uncertainties, the error of evaluation did not exceed the assumed value 20%.

Moreover we can say that:

– if all examined uncertainties of type A are bigger than, or equal to, uncertainties of type B, than the coverage factor $k(\alpha)$ may assume the values of the coverage factor $k_N(\alpha)$ of normal distribution

$$\left.\begin{cases} \sigma_N \geqslant \sigma_R = u_B \\ u_A \geqslant u_B \\ u_{Acl1} \geqslant u_{Bcl} = \Delta_l \\ u_{Acl2} \geqslant u_{Bcl} = \Delta_l \\ u_{Acl3} \geqslant u_{Bcl} = \Delta_l \end{cases}\right\} \Leftrightarrow k(\alpha) = k_N(\alpha) \tag{21}$$

– if all examined uncertainties of type B are bigger than defined uncertainties of type A, respectively, than the coverage factor $k(\alpha)$ may assume the values of the coverage factor $k_N(\alpha)$ of normal distribution or $k_R(\alpha)$ of rectangular distribution. The determined relations decided about it.

$$\left.\begin{cases} u_B = \sigma_R > \sigma_A \\ u_B > u_A \\ \Delta_l = u_{Bcl} > u_{Acl1} \\ \Delta_l = u_{Bcl} > u_{Acl2} \\ \Delta_l = u_{Bcl} > u_{Acl3} \end{cases}\right\} \Leftrightarrow k(\alpha) = \begin{Bmatrix} k_N(\alpha) \\ k_R(\alpha) \end{Bmatrix} \tag{22}$$

However, since it is necessary to establish the defined, required relations in an easy and unequivocal way, we should reject relations whose rate is a function of the number of degree of freedom n.

3. DETERMINING THE DOMINATING COMPONENT OF UNCERTAINTY

It is assumed in metrology that when one component of the sum of two errors is of least ten times bigger than the other one it is considered the dominating component. If we assume that this procedure is based on examining the relations between the numbers which are components of a sum, then for evaluating expanded uncertainty we should examine the relations between the values of components of combined standard uncertainty. However, in many cases these rules are not observed and it is more common to examine the relations betwee limit values of expanded uncertainties. Since the evaluations of type A expanded uncertainties may be different, the results of such comparisons may also be different. Moreover, it results from relations (16), (17) and (18), that the rate of limit expanded uncertainties for each of these three cases considered is greater than the rate of the standard uncertainties. Therefore from the assumption that $u_{Acl}/u_{Bcl}>10$, it follos that uncertainty u_{Bcl}, that is the error introduced by the measuring device, may be neglected. However, the examination of the ratio of standard uncertainties shows that:

$$\frac{u_A}{u_B}=\frac{10}{\sqrt{3\cdot n}}<10, \tag{23}$$

$$\frac{u_A}{u_B}=\frac{10}{\sqrt{3}}<10, \tag{24}$$

$$\frac{u_A}{u_B}=\frac{\sqrt{3}\cdot 10}{t_{q,m}}<10 \tag{25}$$

Therefore, the examination of relations between limit expanded uncertainties cannot be the basis for deciding which uncertainty component is dominating, because it might considerably narrow the limits of the confidence interval.

4. PLANNING THE NUMBER OF TEST MEASUREMENTS ON THE BASIS OF RELATIONS BETWEEN STANDARD UNCERTAINTIES

The appropriate choice of the number of measurements allows to decrease type A uncertainty. Stein's two-step method [2] consists in using the results of the small series of n' measurements to determine the number of measurements n, such that the limit value of type A expanded uncertainty does not exceed the predetermined value d in accordance with the relation:

$$u_{Acl}=k(\alpha)\cdot\frac{\overline{S}}{\sqrt{n}}\leqslant d \tag{26}$$

from this we obtain

$$n\geqslant\frac{k(\alpha)^2\cdot\overline{S}}{d^2} \tag{27}$$

For the small number of measurements the coverage factor $k(\alpha)$ will, in this case, assume the values of standard variable $t_{q,m}$ of the t Student distribution, for $q=1-\alpha$ and for $m=n'-1$. It is necessary to know the results of the first small series of measurements in order to calculate the estimator of the unknown value of standard deviation σ, and to determine the value of the coverage factor $k(\alpha)=t_{q,m}$. We may use the results of the first series of measurements under the assumption that the variation is the same in both series of measurements.

If it follows from relation between the values of type A and type B limit uncertainty that $u_{Acl}>u_{Bcl}$, then we can increase accuracy by increasing the number of measurements up to n, and thus decreasing u_{Acl} in accordance with (26). In such case we usually assume that $d=u_{Bd}$. We can show that in such situation we will obtain a sharp decrease of the value of type A standard uncertainty compared to standard uncertainty of type B, if the number n of measurements is considerably increased. Assuming that the both variations are the same, and substituting n from (26) we get:

$$u_l=\sqrt{u_A^2+u_B^2}=\sqrt{\frac{S}{n}+\frac{\Delta_l^2}{3}} \tag{28}$$

then

$$u_l=\Delta_l\cdot\sqrt{\frac{1}{t_{q,m}^2}+\frac{1}{3}} \tag{29}$$

Since the value of $t_{q,m}$ for a small series of measurements and for $\alpha=0.9973$, is a rule, greater than 3, then $1/t_{q,m}^2 \gg 1/3$. Hence, we must ask to what extend it is worth decreasing standard uncertainty u_A in the case when we cannot further decrease u_B because of the fixed accuracy of the measuring instrument and or of the measurement method. Likewise, to increase exceedingly the number of measurements is neither practical nor cost-effective. In answering the above question we must settle on an adequate relation between u_A and u_B such that:
$u_B/u_A \geqslant b$, where b is an accepted integer. Then:

$$\frac{u_A}{u_B}\geqslant b \tag{30}$$

where b is an accepted integer. Then:

$$u_B=\frac{\Delta_l}{\sqrt{3}}\geqslant b\cdot u_A=b\cdot\frac{\bar{S}}{\sqrt{n}} \tag{31}$$

from this we obtain

$$n\geqslant 3\cdot b^2\cdot\frac{\bar{S}^2}{\Delta_l^2} \tag{32}$$

Without changing the principle of Stein's method, then, we introduce a realistic number of measurements n for a given relation between standard uncertainties of type A and type B.

5. CONCLUSIONS

The author was inspired to compare the relations between the values of various quantities by the absence of any measure which would allow to choose the quantities to be compared. Moreover, the relations used so far might lead to some wrong conclusions particularly when new methods of evaluation of uncertainties are used [2]. The comparative analysis, performed above, leads to the following conclusions:

– If there is significant difference between the values of uncertainties, whose relations are examined, then the convolution of component distributions converges to the distribution of greater uncertainty. In the most critical situation when the values of uncertainties are very close to each other, the error may be considerable much greater than the assumed value of 20% (if the described conditions are not fulfilled). The conditions are different for different relations.

– There are considerable differences when we evaluate the dominating component uncertainty, examining the relations between other uncertainties than standard uncertainties.

– In choosing the number of measurements n with Stein's two-step method, the researcher must settle on the relation between standard uncertainties, determined by the integer b.

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BADANIE WYNIKÓW PORÓWNAŃ MIĘDZY WYBRANYMI NIEPEWNOŚCIAMI

Streszczenie

Przedstawiono pewne sytuacje, w których niezbędne jest dokonywanie porównań między niepewnościami. Wyróżniono sytuacje, w których wynik porównań między niepewnościami może decydowwać: o wyborze odpowiedniej przybliżonej metody oceny niepewności całkowitej, o dominacji standardowych niepewności składowych oraz o planowanej liczności próby. Przedstawiono wyniki badań skutków i różnic w ocenie, spowodowanych wyborem różnych niepewności, między którymi badane są relacje. Jako kryterium wyboru niepewności, które mają być porównywane, przyjęto dokładność uzyskiwanych ocen.